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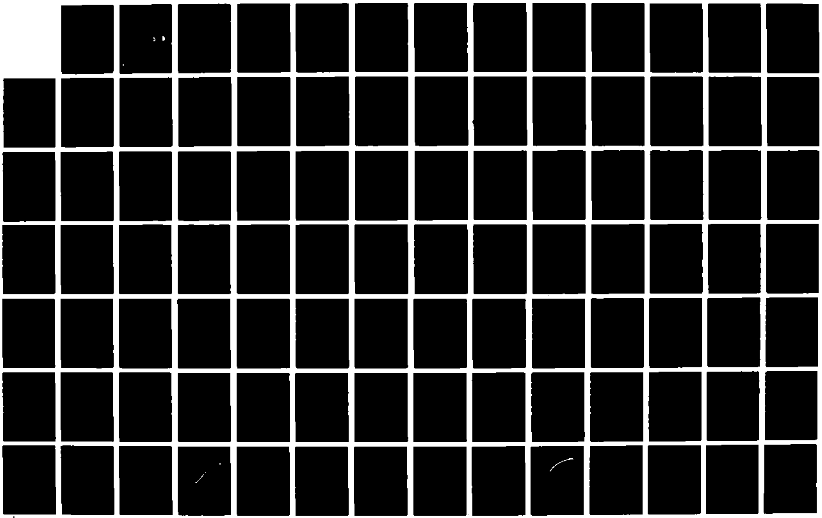

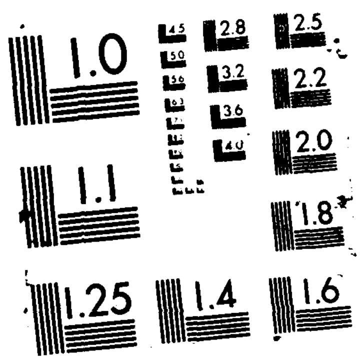
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Abstract of "Steady Waves in a Nonlinear Theory of Viscoelasticity" by Gregory Thomas Warhola, Ph.D., Brown University, May 1988.

This work considers the propagation of steady waves in viscoelastic materials for which the nonlinear strain measure is not necessarily convex. The shape of such a wave is governed by an ordinary nonlinear integro-differential equation having a possibly singular difference kernel. The existence and structure of a solution depends upon the relation of the wavespeed, a parameter in the problem, to two speeds based upon the state of the material ahead of the wave. Solutions are constructed by a monotone iterative scheme which is proven to converge to a unique *solution within restricted* classes of functions depending upon the wavespeed. A simple numerical approximation to the iterative scheme is used to produce graphs of solutions. An algebraic "quasielastic" approximation produces upper bounds on discontinuous (shock and acceleration wave) solutions. For a material such as polymethyl methacrylate (PMMA) having a small power in a power-law model of its compliance, this approximation is found to be useful for accurately predicting the structure of shock solutions.

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Steady Waves in a Nonlinear Theory of Viscoelasticity

by

Gregory Thomas Warhola

B.S., University of Utah, 1980

M.S., Air Force Institute of Technology, 1981

Thesis

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Vita

Gregory Thomas Warhola was born April 29, 1955 in Philadelphia, Pennsylvania. He studied electrical engineering at the University of Utah, earning the degree of Bachelor of Science in 1980. During that year, he was commissioned as a second lieutenant in the United States Air Force and entered the Air Force Institute of Technology for graduate study. There he received the degree of Master of Science in electrical engineering in 1981. He was promoted to captain while assigned to the 544th Strategic Intelligence Wing, Strategic Air Command, Offutt Air Force Base, Nebraska. He entered Brown University in 1984 to study applied mathematics for the degree of Doctor of Philosophy.

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Table of Contents

CHAPTER 1: INTRODUCTION

We are concerned with the propagation of mechanical disturbances in viscoelastic materials. Any such a disturbance in a material is called a *wave* if it has features which are clearly recognizable as it progresses through the material. In a one-dimensional theory where x and t represent place and time, respectively, a wave is ***steady*** if its features are described by a single ***shape*** function, say $w(\theta)$, where $\theta = t - x/V$. The constant V is the ***wavespeed***, the velocity of propagation of the disturbance w.

Transient loading experiments conducted by Schuler [1.1] on the viscoelastic material polymethyl methacrylate (PMMA) have produced waves which appear steady over intervals of observation. In this and other viscoelastic materials having a nonlinear dependence of the stress upon the strain, the tendency for disturbances to dissipate due to time-dependent viscoelastic effects is opposed by the nonlinearity. Shock waves are produced if the loading is sufficiently high. Such waves are mathematically characterized by a discontinuity in the material particle velocity. Kolsky has produced travelling waves in stretched natural rubber [1.2], which waves exhibit a rapid variation in the particle velocity at the wave front.

Early mathematical investigations into the propagation of steady acceleration and shock waves were conducted by Coleman, Gurtin, and Herrera [1.3], in which they assumed e‹istence of such solutions. Acceleration waves are characterized by a discontinuity in some derivative of the particle velocity. Such waves travel at a critical value of wavespeed, with respect to the state of the material ahead of the wave; shocks travel at supercritical speeds. Pipkin [1.4] demonstrated the existence of steady shock and acceleration wave solutions for a specific model of a nonlinear viscoelastic fluid. Furthermore, his treatment includes smooth solutions at subcritical wavespeeds. In an abstract mathematical setting, Greenberg proved the existence of steady shock waves for a broad class of nonlinear viscoelastic materials [1.5]. Greenberg and Hastings [1.6] later studied steady waves in viscoelastic materials in a somewhat less abstract setting. A lengthy description of experimental and mathematical investigations into this subject until 1974 is contained in the work by Nunziato *et al* [1.7].

Our goal is to study the propagation of steady waves in nonlinear viscoelastic materials with yet less abstraction while providing a treatment which is sufficiently general to include useful current models of such materials. This is a purely mechanical treatment; no thermodynamic quantities are considered. In Chapter 2, we provide foundation material for the rest of this work. The equations governing material deformation are presented along with the constitutive equation describing the viscoelastic materials considered. We describe at some length the general nonlinearity we consider, since it includes a departure from the convexity requirement imposed in the earlier treatments [1.5] and [1.6]. We also introduce the *quasi-elastic* approximation to the constitutive equation. We later find this approximation to be accurate enough for description of shock waves in materials like PMMA.

We derive a nonlinear integro-differential equation governing steady waves in Chapter 3. This equation contains the wavespeed as a parameter. After a brief look at steady waves in a linear viscoelastic theory, we prove some general results concerning steady waves in a nonlinear viscoelastic material. From these results, we propose candidate solutions to the problem, which solutions depend upon the relation of the wavespeed to a critical value. In Chapter 4, we prove the existence and uniqueness of these steady wave solutions. We use an iteration scheme to produce monotone sequences of successive approximations to the solution. Our approach is in the spirit of the work of Greenberg and Hastings [1.5]; however, we have relaxed some of their hypotheses on the materials considered. Additionally, we use a different iteration scheme which is amenable to a simple numerical approximation for the construction of graphs of approximate solutions.

In Chapter 5, we consider the detailed structure of steady shock and acceleration waves near the discontinuity. We also discuss the continuity of such solutions as a function of the wavespeed. We show how the information in a known solution at a given wavespeed can be used to obtain the solution for another wavespeed. The numerical approximation to the iteration scheme is introduced. We use it to construct approximate solutions for shock and acceleration waves in materials characterized by exponential and power-law moduli. It is here where we see that the quasi-elastic approximation *is close to the exact solution* for power-law materials having a small power. The graphs of shocks we produce are in good agreement with the experimental results for PMMA obtained by Schuler [1.1].

The final chapter, 6, is devoted to smooth solutions below the critical (acceleration wave) wavespeed. We use a perturbation technique to obtain solutions for waves of infinitesimal amplitude in a viscoelastic solid. We construct solutions via iteration for specific examples involving power-law materials.

CHAPTER 2: NONLINEAR VISCOELASTIC DEFORMATIONS

2.1 Governing Equations.

We consider one-dimensional deformations of an infinite homogeneous nonlinearly viscoelastic body from a (possibly strained) reference configuration in which the body is at rest and its material points (or particles) are identified with their position x on the real line. Let $u(x,t)$ be the displacement at x at the time t of a particle from its reference position. We consider functions u which are continuous and piecewise differentiable in each of their arguments. We write the partial derivatives $\partial u(x,\cdot)/\partial x = u_x(x,\cdot)$ and $\partial u(\cdot,t)/\partial t = u_t(\cdot,t)$. The perturbed particle velocity is

$$v(x,t) = u_t(x,t). \tag{2.1.1}$$

The perturbed strain is given by

$$\varepsilon(x,t) = u_x(x,t). \tag{2.1.2}$$

We normalize the extra stress $\sigma(x,t)$ by the mass density and write the balance of linear momentum as

$$v_t(x,t) = \sigma_x(x,t), \tag{2.1.3}$$

where these derivatives exist. From equations (2.1.1) and (2.1.2), the compatibility condition is

$$\varepsilon_t(x,t) = v_x(x,t). \tag{2.1.4}$$

The following constitutive relation defines the nonlinear viscoelastic materials considered in this work; it is a viscoelastic extension of the relation $\sigma = f(\varepsilon)$ used in nonlinear elasticity. We consider materials for which the stress at the current time t depends on the current strain and the entire previous strain history through the convolution relation

$$\sigma(x,t) = \int_{-\infty}^{+\infty} G(t-\tau)\,df(\varepsilon(x,\tau)). \tag{2.1.5}$$

The integral in equation (2.1.5) is a Stieltjes integral. We describe its properties which are essential for this work in the following section. All of the nonlinearity in the problem is embodied in the strain curve $f(\varepsilon)$, whose graph is local to the reference state such that $f(0) = 0$. Further properties of the curves f that we consider are contained in section 2.3. The function $G(t)$ is the stress relaxation function with units of modulus divided by mass density. Its properties are contained in section 2.5, after a section on ***regularly-varying*** functions. We discuss some properties of the constitutive law in section 2.6.

We note that in a more general treatment one *could consider the constitutive* model

$$\sigma(x,t) = G_e g(\varepsilon(x,t)) + \int_{-\infty}^{+\infty} [(G(t-\tau) - G_e)]\,df(\varepsilon(x,\tau)), \tag{2.1.6}$$

where G_e is a positive constant. In this model the equilibrium elastic stress $G_e g(\varepsilon)$ has a different nonlinear dependence on the strain than does the transient integral term. In cases where the curves f and g are determined experimentally, this approach doubles the data required to model the nonlinearity in the problem. For this work, we **assume** that the nonlinearity is characterized by a single set of these measurements, such that $f = g$, in which case equation (2.1.6) reduces to (2.1.5).

2.2 Properties of the Stieltjes Integral.

Throughout this work we will be concerned only with functions which may be written as the indefinite integral of their derivative, in the generalized sense described below. Such functions have ***bounded variation*** on any finite interval. These properties are enough to ensure the existence of the Stieltjes integral. A proof of its existence and a collection of its properties is contained in the monograph by Widder [2.1]. We illustrate only what is needed herein. Consider, for example, the improper integral:

$$I(\cdot) = \int_{-\infty}^{+\infty} K(\tau, \cdot)\, d\mu(\tau), \tag{2.2.1}$$

where μ is monotone. (If μ is not monotone, it can always be decomposed as $\mu = \mu_+ - \mu_-$, where μ_+ and μ_- are monotone.) If $\mu(t)$ has a piecewise continuous derivative $\mu'(t)$, then $d\mu(t) = \mu'(t)dt$ in equation (2.2.1), which is then evaluated as an ordinary Riemann integral. Thus, if $\mu(t) = t$, then I reduces to the Riemann integral of K. If, however, μ has a jump at t_o given by

$$\Delta\mu(t_o) = \lim_{t \downarrow t_o} \mu(t) - \lim_{t \uparrow t_o} \mu(t), \tag{2.2.2}$$

then

$$I(\cdot) = K(t_o, \cdot)\Delta\mu(t_o) + \int_{-\infty}^{+\infty} K(\tau, \cdot)\mu'(\tau)\, d\tau. \tag{2.2.4}$$

In particular, for a function $J(t)$ which vanishes for $t < 0$, has a finite value J_o at $t = 0$, and whose derivative is given for $t > 0$ by $J'_+(t)$, we have

$$\begin{aligned}\int_{-\infty}^{t} dJ(\tau) &= \int_{0}^{t} dJ(\tau)\\ &= J_o + \int_0^t J'_+(\tau)\, d\tau\\ &= J_o + J(t) - J_o\\ &= J(t).\end{aligned} \tag{2.2.5}$$

Integrals like $\int_0^t dJ(\tau)$ are understood to include integration of the jump in J at an endpoint of the integration interval, if such a jump exists. We use the notation $f_+(t)$ to denote the restriction of f to positive values of t.

2.3 The Strain Curve $f(\epsilon)$.

The strain curve $f(\epsilon)$ is assumed to be a single-valued continuous function of ϵ with a piecewise continuous derivative $f'(\epsilon)$. When the deformations are small we expect that the linear theory of viscoelasticity would apply; this leads us to consider curves f which have the property

$$f(\epsilon) \sim \epsilon, \qquad \epsilon \rightarrow 0. \tag{2.3.1}$$

We limit discussion to functions f which are strictly greater than ϵ, over the range of ϵ considered, when the strain is non-zero:

$$\begin{aligned} f(\epsilon) &> \epsilon, && \epsilon \neq 0 \\ f(\epsilon) &= 0, && \epsilon = 0. \end{aligned} \tag{2.3.2}$$

We do not require f to be globally convex. Instead, we consider the possibly non-convex functions whose graphs have a unique intersection with any ray from the origin, over the range of ϵ considered. For the cases in which we are most interested, ϵ is positive. We require that the equation

$$f(\epsilon) - m\,\epsilon = 0, \qquad m > 1, \tag{2.3.3}$$

have a unique positive solution ϵ, for each value of m. We now show that these properties are sufficient for us to write any f we consider, for positive strain, as

$$f(\epsilon) = m(\epsilon)\,\epsilon, \qquad \epsilon \geq 0, \tag{2.3.4}$$

where $m(\varepsilon)$ is a continuous function, increasing monotonically from $m(0) = 1$, with a piecewise continuous derivative $m'(\varepsilon)$. We already have as an assumption (2.3.3) that ε is uniquely determined by m. Now, $m = f(\varepsilon)/\varepsilon$ uniquely determines m as a function of ε for $\varepsilon > 0$. Furthermore, $m(0) = 1$ follows directly from (2.3.1). Thus, there is a one-to-one relationship between m and ε; $m(\varepsilon)$ is monotone. Property (2.3.2) implies that m is increasing (strictly, since m is one-to-one). The continuity of f requires that $m(\varepsilon)$ be continuous. Since we require $f'(\varepsilon)$ to be piecewise continuous, and since

$$f'(\varepsilon) = m(\varepsilon) + m'(\varepsilon)\,\varepsilon \tag{2.3.5}$$

$m'(\varepsilon)$ must, too, be piecewise continuous.

Conversely, any function f of the form (2.3.4) necessarily has the properties we desire. The continuity of f and the piecewise continuity of its derivative follow from the like properties of m and m' when used in equations (2.3.4) and (2.3.5). Properties (2.3.1) and (2.3.2) are satisfied since $m(0) = 1$ and m is increasing. Finally, (2.3.3) has a unique solution ε since $f/\varepsilon = m(\varepsilon)$ is monotone.

A representative curve $f(\varepsilon)$ is shown in Figure 2.1. From (2.3.4), we have that the slope of the secant from the origin to the point $(\varepsilon, f(\varepsilon))$ is $m(\varepsilon)$. Since $m(\varepsilon)$ is increasing, we see from (2.3.5) that

$$f'(\varepsilon) \geq m(\varepsilon), \qquad \varepsilon > 0, \tag{2.3.6}$$

with equality if and only if $m'(\varepsilon) = 0$; the points having this property cannot be a dense set, since m is strictly increasing.

Equation (2.3.6) implies that f increases monotonically when $\varepsilon > 0$. Additionally, f increases fast enough so that the slope of the secant from the origin to

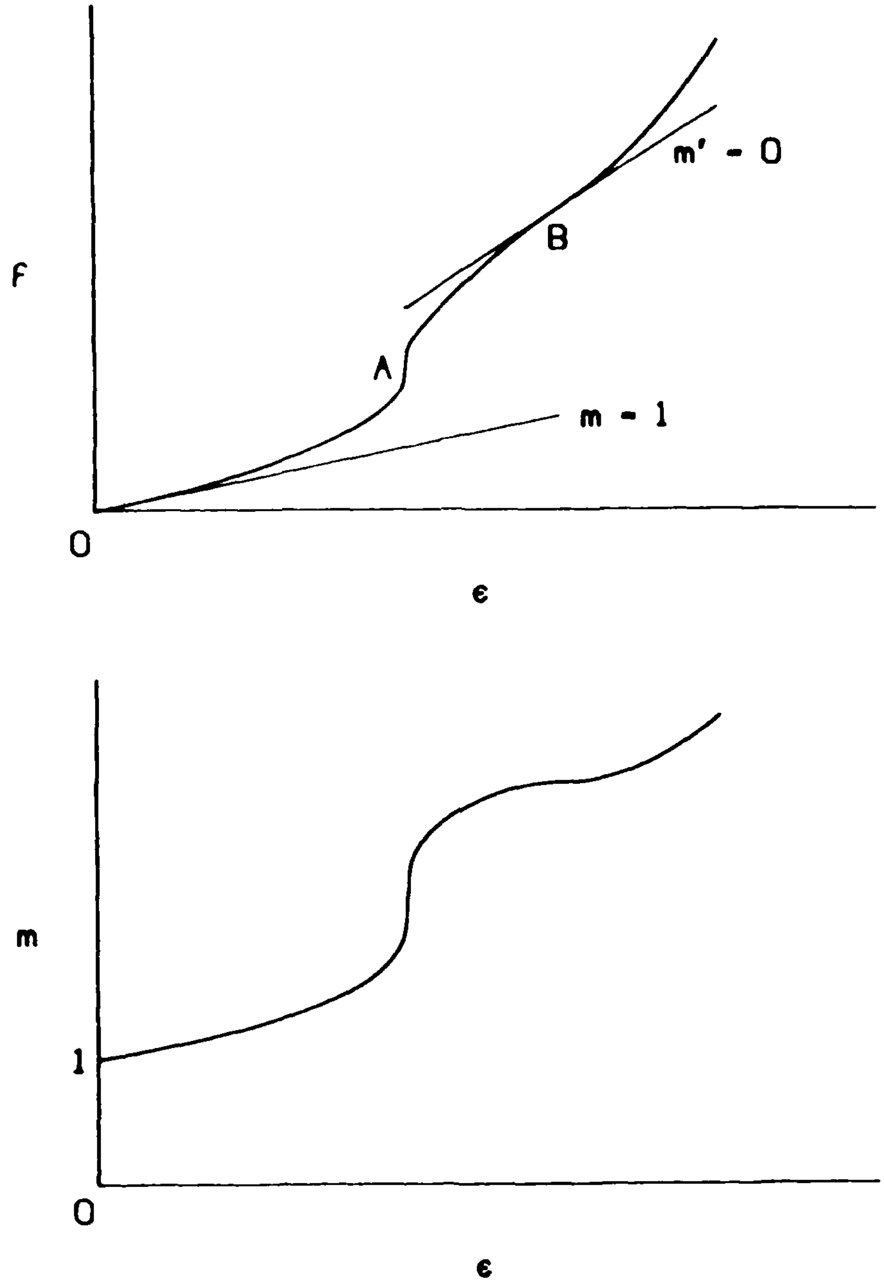

Figure 2.1. A representative strain curve $f(\epsilon)$ (top) and the monotone function $m(\epsilon)$ which generates it (bottom). f is asymptotically linear for small ϵ. At A, the slope of f is infinite. At B, f is tangent to the secant to B from the origin.

any given point on the curve $f(\epsilon)$, is less than the slope of the secant to the given point from any other point on the curve, when ϵ is positive for both points

$$m(\epsilon_1) = \frac{f(\epsilon_1)}{\epsilon_1} < \frac{f(\epsilon_2) - f(\epsilon_1)}{\epsilon_2 - \epsilon_1}, \qquad \epsilon_1 \neq \epsilon_2, \qquad \epsilon_1, \epsilon_2 > 0 \tag{2.3.7}$$

We prove this first for $\epsilon_2 > \epsilon_1$, in which case $m(\epsilon_2) > m(\epsilon_1)$, and thus, $f(\epsilon_2) > m(\epsilon_1)\epsilon_2$. We subtract $f(\epsilon_1) = m(\epsilon_1)\epsilon_1$ from the latter inequality, divide by the positive quantity $\epsilon_2 - \epsilon_1$, and obtain the desired result. On the other hand, when $\epsilon_2 < \epsilon_1$, we have $m(\epsilon_2) < m(\epsilon_1)$. Then $f(\epsilon_2) < m(\epsilon_1)\epsilon_2$ and $-f(\epsilon_2) > -m(\epsilon_1)\epsilon_2$. By adding $f(\epsilon_1) = m(\epsilon_1)\epsilon_1$ to the latter inequality, and dividing by $\epsilon_1 - \epsilon_2$, the proof is complete.

In the proof of (2.3.7), there is the tacit statement that for all $0 < \epsilon < \epsilon_o$, the graph of $f(\epsilon)$ lies below the secant from the origin to the point $(\epsilon_o, f(\epsilon_o))$. We state this separately for future use as:

$$f(\epsilon) < m(\epsilon_o)\epsilon, \qquad 0 < \epsilon < \epsilon_o \tag{2.3.8}$$

Similarly, we have:

$$f(\epsilon) > m(\epsilon_o)\epsilon, \qquad \epsilon > \epsilon_o \tag{2.3.9}$$

We will have occasion to consider ϵ as a function of f. The above properties of f ensure that it has a strictly increasing inverse F, such that $F(f) = \epsilon$. This inverse has properties similar to those of f:

$$\begin{aligned} F(f) &\sim f, \qquad f \to 0, \\ F(f) &< f, \qquad f > 0 \end{aligned} \tag{2.3.10}$$

For each positive $M < 1$, there is a unique non-zero value of f which satisfies

$$F(f) - Mf = 0 \tag{2.3.11}$$

So, we write

$$F(f) = M(f)f, \qquad f > 0, \tag{2.3.12}$$

where $M(f)$ is the slope of the secant from the origin to the point $(f, F(f))$, on the curve $F(f)$. As such, $M(f)$ is the reciprocal of $m(\epsilon)$,

$$M(f) = \frac{1}{m(F(f))}, \qquad f \geq 0, \tag{2.3.13}$$

and is strictly decreasing from $M(0) = 1$. Its rate of decrease is limited by the requirement that F be increasing. From an equation analogous to (2.3.5), M must satisfy

$$|M'(f)| \leq \frac{M(f)}{f}, \tag{2.3.14}$$

with equality only at points not belonging to a dense set. Analogous to equations (2.3.6) through (2.3.9), we have for F:

$$F'(f) \leq M(f), \qquad f \geq 0, \tag{2.3.15}$$

$$\frac{F(f_2) - F(f_1)}{f_2 - f_1} < M(f_1); \qquad f_1 \neq f_2; \qquad f_1, f_2 > 0; \tag{2.3.16}$$

and

$$F(f) \gtrless M(f_o)f \qquad \text{if and only if} \qquad f \lessgtr f_o. \tag{2.3.17}$$

We will sometimes need more detailed information about the strain curve for our results. For example, near the origin, we will assume that f admits the expansion

$$f(\epsilon) = \epsilon + k'\epsilon^\gamma + o(\epsilon^\gamma), \qquad \epsilon \downarrow 0, \tag{2.3.18}$$

for some constants $k' > 0$ and $\gamma > 1$. We will also assume that the inverse F is bounded below

$$F(f) > f - kf^\gamma, \qquad k > 0, \qquad \gamma > 1, \tag{2.3.19}$$

for all values of $f > 0$.

2.4 Regularly-Varying Functions.

The presentation in this section is, for the most part, identical to that given by Pipkin [2.2].

Real material response functions such as $G(t)$ vanish for $t < 0$ and are strictly positive for $t > 0$. Because of the wide variation in both the values of G and the time scales over which these changes take place, they are usually specified by graphs on a doubly-logarithmic scale [2.3]. Let $p(t)$ be the slope on a doubly-logarithmic plot, for $t > 0$, of a differentiable response function $f(t)$:

$$p(t) = d(\ln f(t))/d(\ln t) = tf'(t)/f(t), \tag{2.4.1}$$

where f' is the derivative of f. We note that $p(t) = p$ if $f = ct^p$. In the general case, by integrating (2.4.1) we obtain

$$\ln(f(ct)/f(t)) = \int_1^c p(tx)(dx/x). \tag{2.4.2}$$

Then if $p(t)$ approaches a constant value p, say, as $t \to \infty$, the integral in (2.4.2) approaches $p \ln(c)$, and thus

$$f(ct) \sim c^p f(t), \qquad t \to \infty. \tag{2.4.3}$$

Regularly-varying functions are those functions for which $f(ct)/f(t)$ approaches a finite non-zero limit as $t \to \infty$, and for any such function the limit is necessarily of the form c^p as a function of the parameter c, with $-\infty < p < +\infty$ (see Feller [2.4]). For such a function we write $P(f) = p$ and say that p is the *power* of f.

If equation (2.4.3) holds with $p = 0$ the function is *slowly-varying*. We use $L(t)$ (to suggest a logarithm) to stand for any otherwise unspecified slowly-varying function. Then

$$L(ct) \sim L(t), \qquad t \to \infty. \tag{2.4.4}$$

and any regularly-varying function $f(t)$ can be expressed in the form

$$f(t) = L(t)t^p. \tag{2.4.5}$$

For many purposes L can be treated as if it were constant, even though it may diverge to zero or infinity. For example, powers and products of regularly-varying functions are also regularly-varying, with the obvious exponents. In particular, any function which approaches a non-zero constant value is slowly-varying.

The assumption that $p(t)$ (in equation (2.4.1)) approaches a limit implies not only that f is regularly-varying but also that f' is regularly-varying (unless f is merely slowly-varying). If $p(t) \to p$ then from (2.4.1),

$$\left.\begin{array}{ll} f' \sim pf/t, & p \neq 0 \\ f' = o(f/t), & p = 0 \end{array}\right\} \quad t \to \infty. \tag{2.4.6}$$

The exception for $p = 0$ occurs because when f is approaching a constant, f' may be approaching zero much faster than $1/t$.

On the other hand, when f is regularly-varying its indefinite integral f_1 is always regularly-varying [2.4]. When f_1 diverges as $t \to \infty$,

$$f_1(t) = \int_0^t f(\tau)d\tau \sim \frac{tf(t)}{(p+1)}, \qquad p = P(f) > -1, \qquad t \to \infty, \tag{2.4.7}$$

just as if f were actually a power. When $p < -1$ the integral (2.4.7) approaches a constant and the tail of the integral is regularly-varying:

$$\int_t^\infty f(\tau)d\tau \sim \frac{tf(t)}{|p+1|}, \qquad p < -1, \qquad t \to \infty. \tag{2.4.8}$$

The result analogous to (2.4.6b) is

$$tf(t) = o(f_1), \qquad p = -1, \qquad t \to \infty \tag{2.4.9}$$

The presentation above, following Pipkin [2.2], is useful also for consideration of functions which are regularly-varying at the origin. For, $f(t)$ varies regularly at 0 if and only if $f(1/t)$ varies regularly at ∞ (Feller [2.4]). Thus, these results apply with $t \to \infty$ replaced by $t \to 0$, for functions which vary regularly with power p at the origin. In particular, the change of variables $\tau \to 1/\tau$ in equations (2.4.7) and (2.4.8) returns the same pair of integral results for $t \to 0$.

We add here a result that is needed later for the convolution of two regularly-varying functions. If f and g vanish for $t < 0$, and are regularly-varying as $t \to \infty$ $(t \to 0)$ with powers $P(f) = p$ and $P(g) = q$, then

$$\int_0^t f(t-\tau)g(\tau)d\tau \sim \frac{p!\,q!}{(p+q+1)!} tf(t)g(t); \qquad p, q > -1; \qquad t \to \infty (t \to 0); \tag{2.4.10}$$

where

$$p! = \int_0^\infty t^p e^{-t} dt, \qquad p > -1. \tag{2.4.11}$$

To get this result, we use expressions like (2.4.5) for f and g in the convolution integral and make the change of variables $\tau \to t\tau$. Then, the use of equation (2.4.4) leaves a well-known Beta function whose value is the ratio of factorials in the result.

2.5 Material Response Functions.

In the linear theory of viscoelasticity our stress relaxation function $G(t)$ gives the time history of the stress (per unit mass density) in a viscoelastic material which is subjected to a unit step in strain, ε, at time $t = 0$ (see Pipkin [2.5]). In spite of our normalization with respect to mass density, we will refer to G as the *modulus*. For the nonlinear theory defined by equation (2.1.5) the stress is $G(t)$ when $f(\varepsilon(\cdot,t)) = H(t)$, the Heaviside unit step function defined to be zero for $t < 0$ and one for $t \geq 0$. The modulus is necessarily zero for $t < 0$, since the current stress cannot depend upon the future strain; the input-output relationship between the strain and the stress is *causal*. With this in mind, we write the convolution (2.1.5) as:

$$\sigma(x,t) = \int_{-\infty}^{t} G(t-\tau)\,df(\varepsilon(x,\tau)). \tag{2.5.1}$$

We find it convenient to make operational use of such convolutions and we use the following notation for convolution with respect to the time variable:

$$(f' \star g)(t) = \int_{-\infty}^{+\infty} g(t-\tau)\,df(\tau). \tag{2.5.2}$$

The limits of integration extend to $\pm\infty$ in the general case. We recall that convolution is commutative and associative, and that it also commutes with time differentation:

$$(f \star g)' = f' \star g = f \star g'. \tag{2.5.3}$$

In formal operational use of the convolution, we use whichever of these is convenient at the time. With this notation and these properties of the convolution, we write the nonlinear constitutive law in terms of the modulus, once and for all as:

$$\sigma(x,t) = (G' \star f(\varepsilon))(x,t). \tag{2.5.4}$$

We consider moduli G which have a finite value G_o at $t = 0$ and which decrease to a strictly positive equilibrium value G_e as $t \to \infty$. In fact, we assume that G is *completely monotone* for $t > 0$; i.e., G is positive and decreasing, and it possesses derivatives of all orders which satisfy:

$$(-1)^n G^{(n)}(t) > 0; \qquad n = 0, 1, 2, \ldots; \qquad t > 0. \tag{2.5.5}$$

Since G_e is nonzero, the materials we consider are viscoelastic *solids* [2.5]. It is convenient to allow the *elastic limit*, in which case $G(t) = G_o H(t) = G_e H(t)$.

These properties of G are enough to ensure the existence [2.5] of a *compliance* $J(t)$ which vanishes for $t < 0$, has a completely-monotone derivative $J'_+(t)$ for $t > 0$, and which satisfies:

$$(J' \star G)(t) = H(t). \tag{2.5.6}$$

The compliance takes a finite jump to $J_o = 1/G_o$ at $t = 0$ and increases monotonically thereafter since $J'_+ > 0$. For solids J has an equilibrium value $J_e = 1/G_e$ which it approaches as $t \to \infty$. The units of J are the inverse of those of G; in our case, J has units of $(\text{velocity})^{-2}$. Whenever we write specific forms of G or J, we mean their values for $t > 0$, and leave implicit that they vanish for $t < 0$.

By operating on equation (2.5.4) with J', using (2.5.3) and (2.5.6), we obtain the constitutive law in terms of the compliance:

$$f(\varepsilon(x,t)) = (J' \star \sigma)(x,t), \tag{2.5.7}$$

where we have used the fact that convolution with the Heaviside step function is just an integration. Throughout this work we will use both versions (2.5.4) and (2.5.7) of the constitutive law.

Real material properties might be stated in terms of only one response function, the modulus or the compliance. We now present some relations between the asymptotic behaviour of G and J, for the cases of $t \to 0$ and $t \to \infty$. We sometimes assume that the difference between these functions and their initial or equilibrium limits is regularly-varying. We do so for $t \to 0$ and take for the modulus:

$$G(t) = G_o(1 - \chi_1(t)), \tag{2.5.8}$$

where χ_1 is a positive regularly-varying function:

$$\chi_1(t) = L(t)t^p, \qquad 0 \leq p \leq 1, \qquad t \to 0. \tag{2.5.9}$$

The power p is restricted to these values since G must be completely monotone. Additionally, $\chi_1{}'$ must be completely monotone. Therefore, by equation (2.4.6) (for $p \neq 0$), so must $L(t)t^{p-1}$. Since the product of two completely monotone functions is again completely monotone (Feller [2.4]), any completely monotone slowly-varying $L(t)$ will do. One such function is $|\ln(t)|$. Additionally, for $p \neq 0$, $L \equiv 1$ renders $\chi_1{}'$ completely monotone. If $p = 0$, then L' must be completely monotone.

Similarly, we let the compliance have the form:

$$J(t) = J_o(1 + \chi_2(t)), \tag{2.5.10}$$

and seek to relate χ_2 to χ_1. We use equation (2.5.6) and the relation $J_oG_o = 1$ to obtain a Volterra integral equation for χ_2:

$$\chi_2 = \chi_1 + \chi_1{}' \star \chi_2. \tag{2.5.11}$$

This is asymptotically satisfied with $\chi_2 \sim \chi_1$. For, according to equation (2.4.10), we then have $P(\chi_1{}' \star \chi_2) = 2P(\chi_1) = 2p$, and the convolution is of higher order than χ_1. So,

$$J(t) \sim J_o(1 + \chi_1(t)), \qquad t \to 0, \tag{2.5.12}$$

and we see that G and J are approximately algebraic reciprocals for small t.

For $t \to \infty$, we present some results obtained by Pipkin [2.2]. These were gotten with the use of the Tauberian theorems for Laplace transforms [2.4] to relate the time functions J and G to their transforms in the asymptotic limit. There are two situations to consider which depend upon the time-dependent *apparent viscosity* [2.5]:

$$\eta(t) = \int_0^t \left[G(\tau) - G_e\right] d\tau. \tag{2.5.13}$$

This integral exists for all finite t for the moduli we consider.

We consider first, the cases in which $\eta(t)$ is **not** slowly-varying. Here we restrict attention to moduli for which $G - G_e$ is regularly-varying and write

$$G(t) = G_e\big(1 + \chi(t)\big), \tag{2.5.14}$$

where now,

$$\chi(t) = L(t)t^{-p}, \qquad 0 \leq p < 1. \tag{2.5.15}$$

Cases in which $p \geq 1$ are excluded from this class since we are considering viscosities which are not slowly-varying. Complete monotonicity of G requires that p be non-negative. The result is again that J is approximately the algebraic reciprocal of G:

$$J(t) \sim J_e\big(1 - \chi(t)\big), \qquad t \to \infty. \tag{2.5.16}$$

Now, when $\eta(t)$ is slowly-varying, including all cases for which the integral in (2.5.13) converges at infinity, the response functions have the forms

$$G(t) = G_e\big(1 + \chi_1(t)\big), \quad J(t) = J_e\big(1 - \chi_2(t)\big), \tag{2.5.17}$$

where we do not require χ_1 and χ_2 to be regularly-varying . The relation between χ_1 and χ_2 is:

$$\int_0^t \chi_2(\tau)\, d\tau \sim \int_0^t \chi_1(\tau)\, d\tau = \frac{\eta(t)}{G_e}, \qquad t \to \infty. \tag{2.5.18}$$

This does not imply that $\chi_2 \sim \chi_1$. However, if $\eta(\infty) < \infty$, the relation (2.5.18) becomes an equality in the limit [2.2]. As a simple example of this case, let G and J be given by (2.5.17) for all $t > 0$, and let $\chi_1(t) = \exp\{-t/T\}$. With the use of (2.5.6), we find that $\chi_2(t) = (1/2)\exp\{-t/2T\}$, which is clearly not asymptotic to χ_1. However, both integrate to T in the limit, and the limiting viscosity, $\eta(\infty) = TG_\iota$, is finite.

2.6 The Constitutive Law: Some Properties & the Quasi-Elastic Approximation.

In this section we examine the instantaneous and equilibrium elastic behaviour predicted by our constitutive law for a nonlinearly viscoelastic solid. We associate with this behaviour two wavespeeds which are relevant in this work. We then present an approximation to the viscoelastic constitutive law which treats the material as elastic with a time-varying modulus and compliance. For cases of a monotone input function in the constitutive law and, separately, a non-negative input, we present simple bounds on the output function.

Consider a perturbation strain history at a particular location x which vanishes for $t < 0$, jumps to ε_o at $t = 0^+$, approaches ε_e as $t \to \infty$, and which is continuous, but otherwise arbitrary for $t > 0$. From the constitutive law (2.1.5), the stress is:

$$\sigma(x,t) = f(\varepsilon_o)G(t) + \int_{0^+}^{t} G(t-\tau) f_\tau(\varepsilon(x,\tau))\, d\tau. \tag{2.6.1}$$

For $t < 0$, the stress is zero. Immediately after the jump, it has the value:

$$\sigma(x,0^+) = f(\varepsilon_o)G_o. \tag{2.6.2}$$

On the other hand, in the limit $t \to \infty$ the equilibrium stress from (2.1.5) is:

$$\sigma(x,\infty) = f(\varepsilon_e)G_e. \tag{2.6.3}$$

Equations (2.6.2) and (2.6.3) are the nonlinear elastic stress-strain relations which describe the instantaneous and equilibrium behaviour of our viscoelastic constitutive law. In a theory of wave propagation in nonlinearly elastic materials, the slope of a stress-strain curve at a point (ε, σ) gives the square of the speed of travel (the *wavespeed*) for continuous disturbances having these values of stress and strain. (The slope of a secant of the curve is the square of the speed for discontinuous solutions (shocks), which have jumps corresponding to the values of the stress or strain at the endpoints of the secant.) For our nonlinear theory of viscoelasticity, we will make use of the instantaneous and equilibrium wavespeeds at zero strain [1.5], which are defined, respectively, by:

$$
\begin{aligned}
U_o^2 &= \lim_{\varepsilon \downarrow 0} G_o f'(\varepsilon) = G_o = J_o^{-1} \\
U_e^2 &= \lim_{\varepsilon \downarrow 0} G_e f'(\varepsilon) = G_e = J_e^{-1}.
\end{aligned}
\tag{2.6.4}
$$

With the limiting elastic behaviour as motivation, we now present an approximation which reduces the viscoelastic problem to one of nonlinear elasticity for all time, with time-varying modulus and compliance. We call it the ***quasi-elastic*** approximation; it amounts to approximating the convolution by multiplication in either version of the constitutive law:

$$\sigma_Q(t) = G(t) f(\varepsilon_Q(t)) \simeq (G' \star f(\varepsilon_Q))(t),$$

and (2.6.5)

$$f(\varepsilon_Q(t)) = J(t)\sigma_Q(t) \simeq (J' \star \sigma_Q)(t),$$

where the x-dependence of the stress and strain is left implicit since it has nothing to do with the approximation. We call any such function ε_Q or σ_Q which "solves" a viscoelastic problem based on the use of this approximation a ***quasi-elastic solution***. The relations (2.6.5) become equalities if the material is purely elastic. With this approximation, we treat the material as if this were true at all times by supposing that $G(t)J(t) = 1$ at each value of t.

In cases of a monotonically increasing input to either version of the constitutive law, (2.5.4) or (2.5.7), we can obtain simple bounds on the output function. Consider first (2.5.4) written without the x-dependence as:

$$(G' \star f(\varepsilon))(t) = \int_{-\infty}^{t} G(t - \tau)\, df(\varepsilon(\tau)). \tag{2.6.6}$$

If ε is increasing monotonically, then so is $f(\varepsilon)$, and therefore $df > 0$. Now G satisfies $G_e < G(t - \tau) < G_o$ for $-\infty < \tau < t$, so we have the general bounds:

$$G_e f(\varepsilon(t)) < (G' \star f(\varepsilon))(t) < G_o f(\varepsilon(t)), \qquad -\infty < t < +\infty. \tag{2.6.7}$$

Similarly, from (2.5.7):

$$J_o \sigma(t) < (J' \star \sigma)(t) < J_e \sigma(t), \qquad -\infty < t < +\infty. \tag{2.6.8}$$

If the input function vanishes for $t < 0$, then the lower bound on $G(t - \tau)$ and the upper bound on $J(t - \tau)$ can be made tighter, with the result for $t > 0$:

$$\begin{aligned} G(t)f(\varepsilon(t)) < (G' \star f(\varepsilon))(t) < G_o f(\varepsilon(t)) \qquad & (\varepsilon \equiv 0, t < 0) \\ J_o \sigma(t) < (J' \star \sigma)(t) < J(t)\sigma(t) \qquad & (\sigma \equiv 0, t < 0). \end{aligned} \tag{2.6.9}$$

We note that the inequalities for the bounds in equations (2.6.7) to (2.6.9) become equalities when the material is purely elastic. Observe that for the tighter bounds given by equation (2.6.9), the quasi-elastic approximation serves as a lower bound on the output stress and as an upper bound on the output strain. We also note that the quasi-elastic approximation for **any** monotonically increasing input **always** falls within the general bounds. It is for such inputs a rigourous approximation in that bounds on the error (however loose) are given.

When the input function is known *only to be non-negative*, the upper bound in (2.6.7) and the lower bound in (2.6.8) are still valid. We derive the latter:

$$
\begin{aligned}
(J' * \sigma)(t) &= \int_0^{\infty} \sigma(t - \tau)\, dJ(\tau) \\
&= J_0 \sigma(t) + \int_0^{\infty} \sigma(t - \tau) J'_+(\tau)\, d\tau \\
&> J_0 \sigma(t), \qquad (2.6.10)
\end{aligned}
$$

where we have used equation (2.2.4) to integrate out the jump in J. The result obtains since σ is non-negative and J'_+ is positive since it is completely monotone. The inequality is strict unless $\sigma(\tau)$ vanishes identically for all $\tau \leq t$. Similarly, the upper bound in (2.6.7) results from $G'_+ < 0$.

CHAPTER 3: STEADY WAVES

3.1 Governing Equations.

We study ***steady waves***. These are disturbances of unchanging shape which propagate with a constant speed. In one space dimension,

$$u(x,t) = \hat{u}(t - x/U) \tag{3.1.1}$$

describes a wave of shape $\hat{u}$ which travels with the constant wavespeed U. With $U > 0$, the wave travels "to the right," in the positive x-direction for increasing time t. We call the quantity η defined by:

$$\eta(x,t) = t - x/U \tag{3.1.2}$$

the ***retarded time***.

If the displacement $u(x,t)$ is given by (3.1.1), then the strain ϵ and the velocity v necessarily have the same form:

$$\hat{v}(\eta) = \hat{u}'(\eta) \qquad -U\hat{\epsilon}(\eta), \tag{3.1.3}$$

where the prime denotes differentiation with respect to the argument. To show that this is also true for the stress σ, we use equation (2.5.1) to write

$$\sigma(x,t) = \int_{-\infty}^{t} G(t-\tau)\, df\Big(\hat{\epsilon}(\eta(x,\tau))\Big). \tag{3.1.4}$$

We use the fact that $t - \tau = \eta(x,t) - \eta(x,\tau)$, followed by the change of variables $\eta(x,\tau) \to \tau$ to obtain

$$\sigma(x,t) = \int_{-\infty}^{\eta(x,t)} G(\eta(x,t) - \tau)\, df(\hat{\epsilon}(\tau)). \tag{3.1.5}$$

Therefore, the stress is also a function of the retarded time.

In the rest of this work, the displacement, strain and velocity, as well as the stress will **always** be functions of the retarded time and, for notational simplicity, will be denoted by $u(t)$, etc. The derivatives of these functions will be denoted by $u'(t)$ and so on.

We now derive a nonlinear ordinary integro-differential equation and an initial condition which describe the propagation of steady waves of strain in nonlinearly viscoelastic solids which occupy all of one-dimensional space for all time. As a first step, we use quantities like (3.1.1) in the momentum and compatibility equations (2.1.3) and (2.1.4) to relate the stress to the strain. With the notational conventions above, we obtain

$$\sigma'(t) = U^2 \epsilon'(t). \tag{3.1.6}$$

Recall that our stress and strain are perturbations on a quiet state of the material. In the distant past of physical time, and at places far ahead of the current position of the disturbance, the strain and stress are zero. In terms of the retarded time, this imposes the conditions:

$$\lim_{t \to -\infty} \epsilon(t) = 0 \qquad \text{and} \qquad \lim_{t \to -\infty} \sigma(t) = 0. \tag{3.1.7}$$

Using (3.1.7), we integrate (3.1.6) from $-\infty$ to t and obtain:

$$\sigma(t) = U^2 \epsilon(t). \tag{3.1.8}$$

We now substitute for the stress in (3.1.8) from the constitutive law (2.5.4). The result is:

$$\epsilon(t) = U^{-2}(G' * f(\epsilon))(t). \tag{3.1.9}$$

If instead we use equation (3.1.8) for the stress in the constitutive law (2.5.7), we obtain

$$f(\varepsilon(t)) = U^{2}(J' * \sigma)(t) \tag{3.1.10}$$

Equations (3.1.9) and (3.1.10) are equivalent statements of the integro-differential relation which describes the propagation of steady strain waves in the materials we consider. We call a function $\varepsilon(t)$ a *solution* of our problem if it satisfies equation (3.1.9) and the initial condition (3.1.7). If such a solution exists, the corresponding stress is obtained directly from equation (3.1.8), the velocity from (3.1.3). The displacement is given by

$$u(t) = -U \int_{-\infty}^{t} \varepsilon(\tau)\, d\tau, \qquad t < \infty \tag{3.1.11}$$

For the functions we consider (§§2.1 and §§2.2), this integral is well-defined for all finite t.

3.2 Steady Wave Solutions in Linear Viscoelasticity.

We prepare to study the steady wave problem for nonlinearly viscoelastic materials by first considering the linear problem in which the strain curve is $f(\epsilon) \equiv \epsilon$:

$$\varepsilon(t) = U^2 (J' * \varepsilon)(t), \qquad \epsilon(-\infty) = 0, \qquad -\infty < t < +\infty \tag{3.2.1}$$

This system has continuous solutions

$$\varepsilon(t) = e^{r(t-t_c)}, \qquad -\infty < t < +\infty, \tag{3.2.2}$$

where t_c is an arbitrary time shift and for which the "rise time" $1/r$ satisfies

$$U^2 r \bar{J}(r) = 1, \tag{3.2.3}$$

where

$$r\bar{J}(r) = \int_0^{\infty} e^{-rt}\, dJ(t) \tag{3.2.4}$$

The integral in (3.2.4) is the Laplace-Stieltjes transform of J with the real transform variable r (equivalently $\bar{J}$ is the Laplace transform of J); it is well defined with $r \in (0, \infty)$ for the compliances J we consider (§2.5).

We now show that there is a unique relationship between all positive values of r and the positive values of U which satisfy equation (3.2.3). Using the standard results for the limits of such transforms we have

$$\lim_{r \to \infty} r\bar{J}(r) = \lim_{t \to 0} J(t) = J_o = U_o^{-2}$$
$$\lim_{r \to 0} r\bar{J}(r) = \lim_{t \to \infty} J(t) = J_e = U_e^{-2} \qquad (3.2.5)$$

where we have used equations (2.6.4) to relate these limits to the instantaneous and equilibrium wavespeeds. Now $dJ > 0$ implies that $r\bar{J}(r)$ is (completely) monotone as a function of r [2.4]; thus the relationship between $r \in (0, \infty)$ and $U \in (U_e, U_o)$ is one-to-one. Continuous solutions to (3.2.1) of the form (3.2.2) therefore exist only for these values of U.

We now consider discontinuous solutions to equation (3.2.1). We call such a solution a ***shock*** if it possesses a finite jump discontinuity of the form (2.2.2). If the solution is continuous but its derivative has a jump discontinuity, then we call it an *acceleration wave*; if such a discontinuity appears in a higher order derivative, the solution is a ***higher order acceleration wave***. It is well-known [3.1] that shocks can form in finite time in elastic materials which obey a nonlinear stress-strain relation. Similarly, the equations governing linearly elastic media can support traveling shock solutions provided the discontinuity is in the initial data of the problem; these disturbances travel at the constant wavespeed U, where $\sigma = U^2\epsilon$ is the linear stress-strain relationship. We now show that equation (3.2.1) governing steady waves for the theory of linear viscoelasticity admits non-negative discontinuous solutions among the class of functions possessed of a Laplace transform if and only if the material is purely elastic. Sufficiency is immediate, since the problem reduces to the linear elastic problem. Necessity is obtained in the following. We seek a non-negative solution ϵ which vanishes for $t < 0$, in the class of functions considered. We transform equation (3.2.1), using equation (3.2.4) along with the fact that the transform of the convolution is the product of the transforms of the convolved functions and obtain

$$\bar{\epsilon}(s) = U^{2} s\bar{J}(s)\bar{\epsilon}(s), \tag{3.2.6}$$

where s is the complex transform variable. For $\epsilon \geq 0$ and not identically zero, $\bar{\epsilon}(s)$ is an analytic function of s which is real and positive for all real $s > r_o$, for some $r_o > 0$. Therefore, $s\bar{J}(s) = U^{-2}$ on the segment $s > r_o$. Its unique analytic continuation to the whole complex plane is $s\bar{J}(s) = U^{-2}$; this inverts to give $J(t) = U^{-2}H(t)$. We conclude that non-negative **steady** shocks and acceleration waves which vanish on the interval $(-\infty, 0)$ and possess a Laplace transform exist in the linear theory of viscoelasticity only for purely elastic materials.

3.3 General Characteristics of Nonlinear Steady Waves.

We now consider non-negative solutions to equation (3.1.9) governing nonlinearly viscoelastic steady waves subject to the initial condition (3.1.7). In this section we assume the existence of these solutions. As motivation for our investigation, we first take a geometrical approach to this system. We then derive a set of necessary conditions on a solutions' behaviour. These conditions are presented in a list of general characteristics of nonlinearly viscoelastic steady waves. We call them *properties*. In some cases, they are statements of nonexistence. The proofs of these properties follow their summary. We then use these results along with some results from the previous section to estimate the graphs of solutions.

Consider equation (3.1.9) written in the form:

$$U^2\varepsilon = G' \star f(\varepsilon). \tag{3.3.1}$$

From equation (2.6.7), we have that for monotonically increasing strains ε, the stress $G' \star f(\varepsilon)$ is bounded below by $G_e f(\varepsilon)$ and above by $G_0 f(\varepsilon)$. On a graph of stress vs. strain, this stress always lies between its equilibrium and instantaneous curves. On such a graph, the left side of (3.3.1) is a line with slope U^2 passing through the origin; the wavespeed U is a parameter in the problem. Consider Figure 3.1 and recall the definitions of the equilibrium and instantaneous wavespeeds, U_e and U_o, given in equation (2.6.4). We will see that the value of U in relation to U_e and U_0 determines the nature of the solution ε. From the graph in Figure 3.1 and equation (2.6.3), we expect that a solution has an equilibrium value where the line intersects the lower curve. Similarly, with equation (2.6.2), a solution should exhibit a non-zero instantaneous (discontinuous) response if the line intersects the upper curve other than at the origin. These properties are among those of our desired solutions. We prove that:

1. If $\varepsilon(t)$ is a steady wave solution such that $\varepsilon(t) \to \varepsilon_e > 0$ as $t \to \infty$, then the *equilibrium value*, ε_e, is the unique non-zero solution to

$$f(\varepsilon_e) = U^2 J_e \varepsilon_e. \tag{3.3.2}$$

   If $U > U_e$, then ε_e is strictly increasing as a function of U.

2. There are no non-negative solutions possessed of a non-zero equilibrium value if $U \leq U_e$; hence, the only non-negative monotone solution at these wavespeeds is the trivial solution $\varepsilon(t) \equiv 0$. If $U > U_e$, then a non-trivial non-negative monotone solution has a finite equlibrium value $\varepsilon_e > 0$.

3. For all values of U, if $\epsilon(T) = 0$ for some finite T, then $\epsilon = 0$ for all $t < T$.

4. Any jump discontinuity satisfies the *shock condition*:

$$\frac{\Delta f}{\Delta \varepsilon} = U^2 J_o, \tag{3.3.3}$$

which, for our strain curves f, means that any steady shock travels at a wave-speed $U > U_o$.

5. If $U > U_o$, then any non-trivial non-negative solution ε cannot be globally continuous. It must exhibit one and only one jump discontinuity. This jump, at $t = 0$ (say), is from zero to ε_o, the unique solution to

$$f(\varepsilon_o) = U^2 J_o \varepsilon_o. \tag{3.3.4}$$

For all $t < 0$, $\varepsilon(t)$ is identically zero. As a function of U, ε_o is strictly increasing. Additionally, $\varepsilon_o \leq \varepsilon_e$, with equality if and only if the material is elastic.

6. If $\varepsilon(t)$ is a non-trivial continuous solution which vanishes for, say, $t < 0$, then $U = U_o$ (i.e., U_o is the wavespeed for acceleration waves).

We first prove Property 1. It is convenient to use the equivalent formulation of the problem given in (3.1.10) written as:

$$f(\varepsilon(t)) = U^2 \int_0^\infty \varepsilon(t - \tau)\, dJ(\tau). \tag{3.3.5}$$

If $\varepsilon(t) \to \varepsilon_e > 0$ as $t \to \infty$, then we obtain (3.3.2) as the limiting form of (3.3.5), since $\int_0^\infty dJ(\tau) = J_e$, according to (2.2.5). From the properties of f (§§2.3), non-zero solutions to (3.3.2) are unique and increase strictly with U when $U^2 J_e > 1$, i.e., when $U > U_e$. □

We remark that one could also pose our problem as a search for steady waves having a given non-zero equilibrium value, since for a given a compliance J and a particular $\varepsilon_e > 0$, the squared wavespeed U^2 is uniquely determined from (3.3.2).

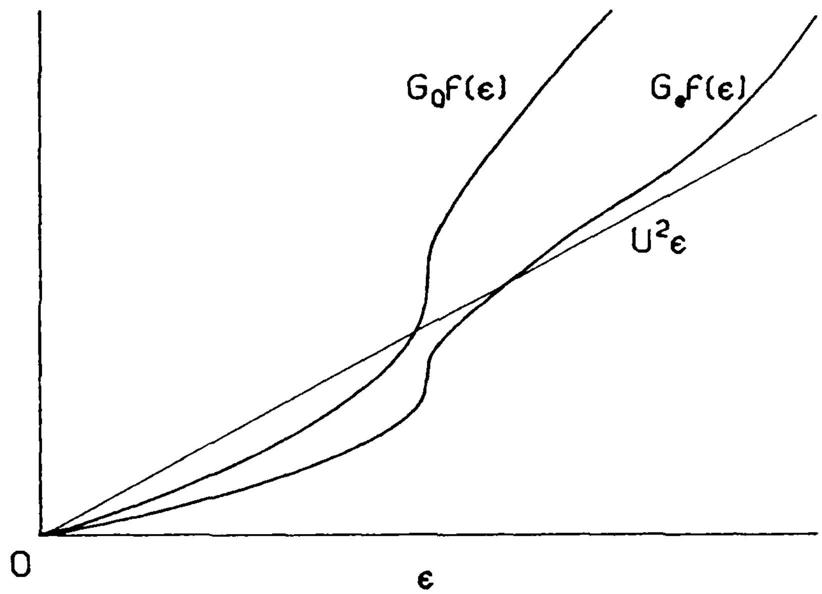

Figure 3.1. The instantaneous and equilibrium stress-strain curves and the line $U^2\varepsilon$.

To prove Property 2, we observe that the unique equilibrium strain from equation (3.3.2) is zero when $U \leq U_e$ for our strain curves f. Thus, there are no non-negative solutions with non-zero ε_e at these wavespeeds. Since a solution ε vanishes as $t \to -\infty$, it can be monotone and vanish as $t \to +\infty$ if and only if it is identically zero. Now, suppose that $\varepsilon(t)$ is a non-trivial non-negative monotone solution to (3.3.5) for some value of $U > U_e$. We use the upper bound on the convolution given in (2.6.8) to write

$$f(\varepsilon(t)) < U^2 J_e \varepsilon(t). \tag{3.3.6}$$

Recall (§§2.3) that $m(\varepsilon) = f(\varepsilon)/\varepsilon$ is strictly increasing from $m(0) = 1$. As such, it has a strictly increasing inverse m^{-1}, defined for $m \geq 1$, such that $\varepsilon = m^{-1}(m)$, with $m^{-1}(1) = 0$. We use this in equation (3.3.6) to obtain

$$\varepsilon(t) < m^{-1}(U^2 J_e) \qquad \text{for all } t. \tag{3.3.7}$$

If $\varepsilon(t)$ is a non-trivial non-negative monotone solution, equations (3.1.7) and (3.3.7) imply the existence of a finite limit $\varepsilon_e > 0$ for $t \to \infty$. □

Property 3 follows by considering $\varepsilon(T) = 0$ for some finite T. The integral in (3.3.5) must vanish; with $dJ > 0$ and for ε non-negative, we conclude that $\varepsilon(t)$ must vanish for all $t < T$. □

We now prove Property 4. Suppose ε has a countable number of jump discontinuities $\Delta\varepsilon_i = \varepsilon(t_i^+) - \varepsilon(t_i^-)$, for $i = 1, 2, \ldots$, where the t_i are labeled in increasing order, and the superscripts + and − denote limits to the point from above and below, respectively. *Using equation* (2.5.3), *we write* (3.3.5) *as:*

$$f(\varepsilon(t)) = U^2 \int_{-\infty}^{t} J(t-\tau)\, d\varepsilon(\tau). \tag{3.3.8}$$

Consider the k-th jump; for $t < t_k$, we have from (3.3.8):

$$f(\varepsilon(t)) = U^2 \sum_{i<k} J(t - t_i)\Delta\varepsilon_i + U^2 \int_{-\infty}^{t} J(t-\tau)\varepsilon'(\tau)\, d\tau, \quad t < t_k, \tag{3.3.9}$$

for which we recall that J is zero when its argument is negative. Similarly,

$$f(\varepsilon(t)) = U^2 \sum_{i\leq k} J(t - t_i)\Delta\varepsilon_i + U^2 \int_{-\infty}^{t} J(t-\tau)\varepsilon'(\tau)\, d\tau, \quad t_k < t < t_{k+1} \tag{3.3.10}$$

In the limit as $t \uparrow t_k$ in (3.3.9) we have that

$$f(\varepsilon(t_k^-)) = U^2 \sum_{i<k} J(t_k^- - t_i)\Delta\varepsilon_i + U^2 \int_{-\infty}^{t_k^-} J(t_k^- - \tau)\varepsilon'(\tau)\, d\tau, \tag{3.3.11}$$

and as $t \downarrow t_k$ in (3.3.10):

$$f(\varepsilon(t_k^+)) = U^2 J_o \Delta\varepsilon_k + U^2 \sum_{i<k} J(t_k^+ - t_i)\Delta\varepsilon_i + U^2 \int_{-\infty}^{t_k^+} J(t_k^+ - \tau)\varepsilon'(\tau)\, d\tau, \tag{3.3.12}$$

where we have used $J_o = J(0^+)$. We subtract (3.3.11) from (3.3.12), and use the continuity of J for $t > 0$ to cancel the summations. The integrals also cancel, since the integrands differ only at the single point t_k. We are left with:

$$U^2 J_o = \frac{\Delta f_k}{\Delta \varepsilon_k}, \tag{3.3.13}$$

which is the desired shock condition for the k-th jump discontinuity. Geometrically, $\Delta f / \Delta \varepsilon$ is the slope of the secant on the curve $f(\varepsilon)$ connecting the values of ε which comprise the jump. Now when both ε^+ and ε^- are positive, we have from equation (2.3.7) that this secant's slope is larger than one, since $m > 1$. If one of ε^+ or ε^- is zero (say ε^-), then $\Delta f / \Delta \varepsilon$ is the slope of the secant from the origin to the point $(\varepsilon^+, f(\varepsilon^+))$, and this slope is also greater than one for non-zero ε^+. Therefore, $U^2 J_o$ is strictly larger than one. This is equivalent to $U > U_o$ which was to be proven for any shock wave. □

To prove Property 5, we first show that if ε has non-zero values for $U^2 J_o > 1$, then there is a "forbidden set" of values which it cannot have. We fix the current time t and assume there is a non-negative function ε which satisfies equation (3.3.5) and whose current value $\varepsilon(t)$ is positive. We integrate out the jump in J and write:

$$f(\varepsilon(t)) - U^2 J_o \varepsilon(t) = U^2 \int_0^\infty \varepsilon(t-\tau) J'_+(\tau)\, d\tau. \tag{3.3.14}$$

Since the integral in (3.3.14) is non-negative, we have

$$f(\varepsilon(t)) - U^2 J_o \varepsilon(t) \geq 0. \tag{3.3.15}$$

For $U^2 J_o > 1$ and for our functions f, relation (3.3.15) is satisfied as an equality by a unique value ε_o, according to equation (2.3.3). As a function of U, ε_o is increasing due to the the characterization of f given in (2.3.4). From (2.3.9), the left side of (3.3.15) is positive for $\varepsilon > \varepsilon_o$. Values of $\varepsilon \in (0, \varepsilon_o)$ are forbidden since they render the left side of (3.3.15) negative, in view of (2.3.8). Therefore, at any time t, either $\varepsilon(t) = 0$ or $\varepsilon(t) \geq \varepsilon_o$ in (3.3.14), when $U^2 J_o > 1$. We conclude that the solution is not globally continuous (unless it is identically zero); it must jump from zero in order that it have non-zero values. By Property 4, this jump must be to the value ε_o. If $\varepsilon(t) = \varepsilon_o$ in equation (3.3.14) when $t = 0^+$, then the integral must vanish at 0^+. For non-negative solutions ε, we must have $\varepsilon(\tau) = 0$ for all $\tau < 0$. Furthermore, this jump discontinuity is the only one possible, since equation (2.3.7) implies that a jump originating from and ending at valt es of ε not in the forbidden set must travel at a wavespeed greater than the value of U which produced the jump to ε_o. Such another value for the wavespeed would contradict Property 4; any other such jump cannot belong to the same steady wave. Finally, it is a trivial consequence of Property 1 and the characterization of f given in (2.3.4) that $\varepsilon_o \leq \varepsilon_e$. If the material is elastic, then $J_o = J_e$ renders ε_o and ε_e identical and this is not possible if the material is not elastic. □

To prove Property 6, we use the assumed properties of ε to choose a sequence of times $\{t_k\}_{k=1}^{\infty}$ with $t_k \downarrow 0$ such that, for each k:

$$\varepsilon(t_k) = \max_{\tau \leq t_k} \varepsilon(\tau). \tag{3.3.16}$$

From equation (3.1.10), we have:

$$f(\varepsilon(t_k)) = U^2 (J' \star \varepsilon)(t_k). \tag{3.3.17}$$

Since ε vanishes for $t < 0$, equations (2.6.9) and (2.3.4) imply that

$$m(\varepsilon(t_k)) \leq U^2 J(t_k), \tag{3.3.18}$$

with equality only if the material is purely elastic. In the limit as $t_k \downarrow 0$ (and therefore as $\varepsilon \downarrow 0$), we have

$$1 = m(0) \leq U^2 J_o = (U/U_o)^2, \tag{3.3.19}$$

with the use of equation (3.2.5). Since ε is assumed continuous, Property 5 implies that U is not greater than U_o; hence, $U = U_o$. □

We now use the above properties to estimate the graphs of candidate steady wave solutions. For $U > U_o$, a candidate solution is a shock which has a single jump at $t = 0$ and continuously approaches the appropriate equilibrium value. At $U = U_o$, we consider a limiting case of shock solutions as the parametric wavespeed is decreased to U_o to obtain a candidate acceleration wave (or a higher order acceleration wave) which vanishes for $t \leq 0$. For $U_e < U < U_o$ a candidate solution is continuous and increases from zero at negative infinity to ε_e as $t \to \infty$. Since ε must be vanishingly small as $t \to -\infty$, we expect it to be asymptotic to a solution of the linearized equation obtained by setting $f(\varepsilon) \equiv \varepsilon$. From the previous section, we know that there are exponential solutions for these values of U. Thus, our candidate solution for the nonlinear problem is asymptotically exponential for $t \to -\infty$. The graphs of these functions for all values of U are shown in figure 3.2.

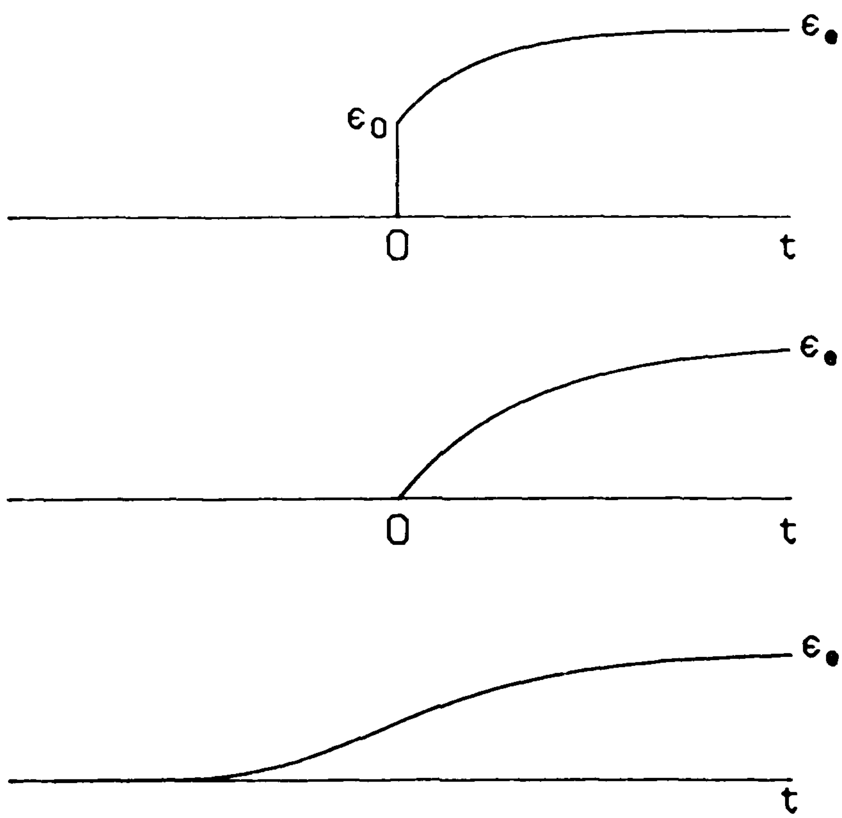

Figure 3.2. Candidate steady wave solutions. From the top: the shock for $U > U_o$, the acceleration wave for $U = U_o$, and the continuous solution for $U_e < U < U_o$.

3.4 The Quasi-Elastic Solution.

In section 2.6 we presented the quasi-elastic approximation to the constitutive law. We use it herein to obtain the quasi-elastic solution ε_Q for nonlinearly visco-elastic steady strain waves. After we examine its characteristics, we discuss its usefulness as an approximation to steady wave solutions.

In terms of the compliance, ε_Q satisfies the following algebraic equation at each t:

$$f(\varepsilon_Q(t)) = U^2 J(t)\varepsilon_Q(t). \tag{3.4.1}$$

We prove these properties of ε_Q:

1. For $U^2J(t) > 1$, there is a unique non-zero quasi-elastic solution ε_Q. If $U^2J(t) \leq 1$, then $\varepsilon_Q(t) = 0$.

2. The quasi-elastic solution has the correct equilibrium value and, for $U > U_o$, the correct instantaneous value.

3. ε_Q is piecewise continuously differentiable and, where it is non-zero, it is strictly increasing.

4. The quasi-elastic solution is an upper bound on monotone solutions to equation (3.1.10) for nonlinearly viscoelastic steady waves which vanish for $t < 0$ (say).

5. If ε_{Q_2} and ε_{Q_1} are the quasi-elastic solutions corresponding to wavespeeds U_2 and U_1 for which $U_2 > U_1$, then $\varepsilon_{Q_2}(t) \geq \varepsilon_{Q_1}(t)$ with equality only at those times where $\varepsilon_{Q_2}(t) = \varepsilon_{Q_1}(t) = 0$.

The existence and uniqueness of ε_Q (Property 1) follow from the properties of our strain curves f. □

For Property 2, the limiting cases of (3.4.1) produce the results in equations (3.3.2) and (3.3.4) for viscoelastic steady waves. □

To prove Property 3, we recall from section 2.3 that $m(\varepsilon) = f(\varepsilon)/\varepsilon$ is piecewise continuously differentiable. Its inverse, $m^{-1}(m)$, defined for $m \geq 1$ is also piecewise continuously differentiable. Thus, from equation (3.4.1) we may express ε_Q directly as:

$$\varepsilon_Q(t) = m^{-1}(U^2 J(t)) \qquad \text{for} \qquad U^2 J(t) \geq 1. \tag{3.4.2}$$

The differentiability and monotonicity properties of m^{-1} and J imply these properties of ε_Q. □

To prove Property 4, we recall from equation (2.6.9) that the quasi-elastic approximation is an upper bound on the constitutive law for monotone input functions which vanish for $t < 0$. Thus, from equation (3.1.10), we have for a monotone viscoelastic solution ε which vanishes for $t < 0$:

$$f(\varepsilon(t)) < U^2 J(t)\varepsilon(t). \tag{3.4.3}$$

However, equation (3.4.1) implies that $U^2 J(t) = m(\varepsilon_Q(t))$, in the notation of section 2.3, and thus,

$$f(\varepsilon(t)) < m(\varepsilon_Q(t))\varepsilon(t). \tag{3.4.4}$$

Using equations (2.3.8) and (2.3.9), we conclude that $\varepsilon < \varepsilon_Q$. □

The proof of Property 5 follows from equation (3.4.2) and the monotonicity of m^{-1}. □

We note that the graphs of ε_Q, determined by m^{-1} and J, look much like those of our candidate steady wave solutions shown in figure 3.2. If $U > U_o$, then ε_Q is a shock which has a jump at $t = 0$, from zero to $m^{-1}(U^2 J_o)$, from which it increases monotonically to ε_e. If $U = U_o$, then ε_Q is a continuous acceleration wave (of possibly higher order) which departs from zero at $t = 0$ and increases monotonically to its equilibrium value. For the cases $U_e < U < U_o$, ε_Q is a shifted version of the above acceleration wave; it is zero until some time $t_o > 0$ which satisfies $U^2 J(t_o) = 1$. It is obvious that for this latter range of U, such a quasi-elastic solution cannot be a global bound for the viscoelastic solution which never vanishes on an interval.

The quasi-elastic solution is attractive for a number of reasons. As it is the solution to an algebraic relation, it is easily computed. For special forms of the strain curve f, ε_Q can be expressed exactly in terms of parameters of f and the compliance J. If f and J are given as interpolators of data, $\varepsilon_Q(t)$ can be determined at any particular time t from a numerical nonlinear equation solver.

The quasi-elastic solution provides an upper bound on the stress in a material experiencing monotone steady wave motion; for, the stress is proportional to the strain by equation (3.1.8), and ε_Q is an upper bound on the strain (Property 4). In the next chapter, we show how a sequence of increasingly tighter upper bounds on the solution ε can be obtained from ε_Q. Moreover, for some shock solutions ($U > U_o$), the quasi-elastic solution can be a rather good approximation in itself. We have shown in this section (Property 2), that ε_Q has the correct limiting values at $t = 0$ and for $t \to \infty$. In Chapter 5, we show further that, for shock solutions, ε_Q is asymptotically correct, as $t \downarrow 0$.

CHAPTER 4: EXISTENCE & UNIQUENESS THEOREMS

4.1 Prelude.

In this chapter, we prove the existence of the kinds of steady wave solutions we chose as candidates in Chapter 3 and we examine the uniqueness of such solutions among certain classes of functions. The proofs we present for the existence theorems are **constructive**. We can, therefore, use the method of proof to obtain a numerical approximation to the solution. We build solutions from monotonically convergent sequences of functions which result from an iteration scheme. Although monotone convergence is generally slow, it has the nice property that each member of a convergent non-increasing (non-decreasing) sequence of functions is an upper bound (lower bound) on the limit function. We make these notions of monotone convergence precise in the next section. In the following sections, we present the iteration scheme and its properties, and we prove the main theorems on existence and uniqueness of steady wave solutions.

4.2 Monotone Convergence & Bounding Sequences.

We denote by $\{f_n\}_{n=1}^{\infty}$ a sequence of real-valued functions defined on some subset of the real line. Consider such a sequence with members which satisfy:

$$f_1(t) \leq f_2(t) \leq \ldots \leq f_n(t) \leq \ldots \leq f_U(t), \tag{4.2.1}$$

at each t, for some finite real-valued function f_U (subscript U for "upper bound"), which is also defined where the f_n are defined. At a fixed value of t, the sequence $\{f_n(t)\}_{n=1}^{\infty}$ is a non-decreasing sequence of numbers bounded above by the number $f_U(t)$. Such a sequence converges to a number we denote by $f(t)$. As t varies, a real-valued function f is defined on the set where the f_n are defined. This limit function is the *pointwise* limit of the sequence $\{f_n\}_{n=1}^{\infty}$; it is bounded below by each member of this sequence and above by f_U for all values of t where it is defined.

$$f_1 \leq f_2 \leq \quad \leq f < f_U \tag{4.2.2}$$

Naturally, an analogous result holds for non-increasing sequences bounded below by a lower bound function f_L.

4.3 The Iteration Scheme.

A standard technique for the solution of integral equations [4.1] is *iteration*. The equation to be solved is arranged as an input-output system into which an initial guess for the solution is fed. The resulting output function is then used as a new input to the system and this process is repeated, generating a sequence of functions. If this sequence converges to a function which satisfies the original equation and any associated side conditions, the limit function is then a solution of the problem. We apply this technique to construct a solution for the system of equations (3.1.10) and (3.1.7) governing the propagation of nonlinearly viscoelastic steady waves. We recall from section 2.3, that the strain curve f has an inverse F. We use this to write (3.1.10) as

$$\epsilon(t) = F\Big(U^2(J' \star \epsilon)(t)\Big). \tag{4.3.1}$$

From this we define the input-output system for use in iteration:

$$\epsilon_{\text{out}}(t) = F\Big(U^2(J' \star \epsilon_{\text{in}})(t)\Big). \tag{4.3.2}$$

The iteration scheme is

$$\varepsilon_{n+1}(t) = F\left(I^{-2}(J' \star \varepsilon_n)(t)\right), \qquad n = 1, 2, \tag{4.3.3}$$

We proceed with the proofs of a series of lemmas related to the iteration scheme which will lead us to the main theorems in the succeeding sections. Throughout, we denote by τ the running time variable and let t be the current time. We begin with a pair of lemmas for the convolution $J' \star \varepsilon$.

Lemma 4.3.1. *If ε is monotone, then $J' \star \varepsilon$ is monotone.*

Proof. We have

$$(J' \star \varepsilon)(t) = \int_0^\infty \varepsilon(t - \tau)\, dJ(\tau) \tag{4.3.4}$$

Since $dJ > 0$, the monotonicity of ε implies that of the integral. □

Lemma 4.3.2. *If $\varepsilon_2(\tau) \geq \varepsilon_1(\tau)$ for all $\tau \leq t$, with ε_2 not identically equal to ε_1 on a set of positive measure, then $(J' \star \varepsilon_2)(t) > (J' \star \varepsilon_1)(t)$.*

Proof:

$$\begin{aligned}(J' \star \varepsilon_2)(t) - (J' \star \varepsilon_1)(t) &= (J' \star (\varepsilon_2 - \varepsilon_1))(t) \\ &= \int_0^\infty \left(\varepsilon_2 - \varepsilon_1\right)(t - \tau)\, dJ(\tau) \\ &> 0,\end{aligned}$$

since $dJ > 0$ and, by the hypotheses, the difference $\varepsilon_2 - \varepsilon_1$ contributes at least a finite positive amount to the integral. □

Now for the input-output system, we have the following lemmas.

Lemma 4.3.3. *(a) If in equation (4.3.2) ε_{in} is monotone, then ε_{out} is monotone. (b) If ε_{in} is continuous except for a finite jump discontinuity at $t = 0$, then ε_{out} has a finite jump discontinuity at $t = 0$ and is otherwise continuous.*

Proof: (a) The monotonicity of ε_{out} follows directly from the monotonicity of both F and $J' \star \varepsilon_{\text{in}}$. Statement (b) follows from the continuity of F, the continuity of J for $t > 0$, and equation (2.2.4) for the Stieltjes integral. □

Lemma 4.3.4. *If $\varepsilon^a_{\text{in}}$ and $\varepsilon^b_{\text{in}}$ are not identical and $0 \le \varepsilon^a_{\text{in}}(\tau) \le \varepsilon^b_{\text{in}}(\tau)$ for all $\tau \le t$, then $\varepsilon^a_{\text{out}}(t) < \varepsilon^b_{\text{out}}(t)$.*

Proof: From the hypotheses and Lemma 4.3.2, we have

$$(J' \star \varepsilon^a_{\text{in}})(t) < (J' \star \varepsilon^b_{\text{in}})(t). \tag{4.3.5}$$

Since F is strictly increasing,

$$F\Big(U^2(J' \star \varepsilon^a_{\text{in}})(t)\Big) < F\Big(U^2(J' \star \varepsilon^b_{\text{in}})(t)\Big), \tag{4.3.6}$$

and the result follows from (4.3.2) and (4.3.6). □

With the next pair of lemmas, we examine the output function obtained when the input is related to the quasi-elastic *solution presented in section 3.4. The first* provides sufficient conditions under which the output function is strictly less than the input function.

Lemma 4.3.5. *If ε_{in} is monotone, vanishes identically for $t < 0$, and satisfies $\varepsilon_{\text{in}}(\tau) \ge \varepsilon_Q(\tau)$ for all $\tau \le t$, then $\varepsilon_{\text{out}}(t) < \varepsilon_{\text{in}}(t)$, for $t > 0$, when the material is not purely elastic.*

Proof: By equation (2.3.9), we have for $\varepsilon_{\text{in}} \ge \varepsilon_Q$:

$$f(\varepsilon_{\text{in}}) \ge m(\varepsilon_Q)\varepsilon_{\text{in}}. \tag{4.3.7}$$

We recall from (3.4.1), that $m(\varepsilon_Q(t)) = U^2 J(t)$, so we have,

$$\begin{aligned} f(\varepsilon_{\text{in}}(t)) &\ge U^2 J(t)\varepsilon_{\text{in}}(t) \\ &> U^2(J' \star \varepsilon_{\text{in}})(t), \end{aligned} \tag{4.3.8}$$

where the strict inequality is obtained from the bound in (2.6.9) for materials which are not purely elastic. Using F, the strictly increasing inverse of f, and the definition of ε_{out}, we obtain:

$$\varepsilon_{\text{in}}(t) > F\Big(U^2(J' \star \varepsilon_{\text{in}})(t)\Big) = \varepsilon_{\text{out}}(t), \tag{4.3.9}$$

which is the stated result. □

The next lemma gives sufficient conditions for the quasi-elastic solution to be a strict upper bound on the output function.

Lemma 4.3.6. *If ε_{in} vanishes for $t < 0$, is monotone, and satisfies $\varepsilon_{\text{in}}(\tau) \leq \varepsilon_Q(\tau)$ for all $\tau \leq t$, then $\varepsilon_{\text{out}}(t) < \varepsilon_Q(t)$, for $t > 0$, when the material is not purely elastic.*

Proof: By Lemma 4.3.2 and the bound in (2.6.9), we have the following inequalities:

$$(J' \star \varepsilon_{\text{in}})(t) < (J' \star \varepsilon_Q)(t) < J(t)\varepsilon_Q(t). \tag{4.3.10}$$

So,

$$\varepsilon_{out}(t) = F\Big(U^2(J' \star \varepsilon_{in})(t)\Big) < F\Big(U^2 J(t)\varepsilon_Q(t)\Big) = \varepsilon_Q(t) \tag{4.3.11}$$

by the monotonicity of F and the definition of ε_Q in equation (3.4.1). □

Analogous to the results of the two previous lemmas, we can provide sufficient conditions for which the output function is strictly greater than the input function and, separately, conditions for which we have a strict lower bound on the output. We consider input functions which are related to ε_L (subscript L for "lower bound"), which we define as:

$$\varepsilon_L(t) = \varepsilon_o H(t), \tag{4.3.12}$$

where ε_o is the unique value to which the solution jumps at $t = 0$ (if it does so), given in equation (3.3.4), and $H(t)$ is the Heaviside step function. The next lemma validates ε_L as a lower bound.

Lemma 4.3.7. *If ε_{in} vanishes for $t < 0$, is monotone, and satisfies $\varepsilon_{\text{in}}(\tau) \geq \varepsilon_L(\tau)$ for all $\tau \leq t$, then $\varepsilon_{\text{out}}(t) > \varepsilon_L(t)$, for $t > 0$, when the material is not purely elastic.*

Proof: From Lemma 4.3.2 we have

$$(J' \star \varepsilon_{\text{in}})(t) \geq (J' \star \varepsilon_L)(t), \tag{4.3.13}$$

with equality if and only if ε_{in} and ε_L are identical. Then, since convolution with the Heaviside step function is just an integration:

$$\begin{aligned}(J' \star \varepsilon_{\text{in}})(t) &\geq J(t)\varepsilon_o \\ &> J_o\varepsilon_o H(t),\end{aligned} \tag{4.3.14}$$

for materials which are not purely elastic. Thus,

$$\varepsilon_{\text{out}}(t) = F\Big(U^2(J' \star \varepsilon_{\text{in}})(t)\Big) > F\Big(U^2 J_o\varepsilon_o H(t)\Big) = \varepsilon_L(t), \tag{4.3.15}$$

from the definitions of ε_{out}, ε_o, and ε_L, and the monotonicity of F. $\square$

Lastly, for the output to be greater than the input, we have

Lemma 4.3.8. *If ε_{in} vanishes for $t < 0$, is monotone, and not identically zero for $t > 0$, and satisfies $\varepsilon_{\text{in}}(\tau) \leq \varepsilon_L(\tau)$ for all $\tau \leq t$, then $\varepsilon_{\text{out}}(t) > \varepsilon_{\text{in}}(t)$, where ε_{in} is non-zero, when the material is not purely elastic.*

Proof: We use the monotonicity of f and equation (2.3.8) to write:

$$f(\varepsilon_{\text{in}}) \leq f(\varepsilon_o) = m(\varepsilon_o)\varepsilon_{\text{in}}, \qquad \tau \leq t, \tag{4.3.16}$$

with equality if and only if ε_{in} is identical to ε_L. Now, $m(\varepsilon_o) = U^2 J_o$, and by equation (2.6.9) we have $J_o\varepsilon_o H(t) < (J' \star \varepsilon_{\text{in}})(t)$, when the material is not purely elastic. Therefore,

$$f(\varepsilon_{\text{in}}(t)) < U^2(J' \star \varepsilon_{\text{in}})(t). \tag{4.3.17}$$

Using the monotonicity of F and the definition of ε_{out} we obtain:

$$\varepsilon_{\text{in}}(t) < F\left(U^2(J' \star \varepsilon_{\text{in}})(t)\right) = \varepsilon_{\text{out}}(t), \tag{4.3.18}$$

which is the stated result. □

4.4 $U > U_o$.

We construct sequences of functions using the interation scheme (4.3.3) for wavespeeds $U > U_o$. We define an ***upper bound*** sequence to be a monotone sequence of functions, each member of which is an upper bound on all succeeding members. Similarly, a ***lower bound*** sequence is one for which each member is a lower bound on all succeeding terms.

We first construct an upper bound sequence. We fix a value of $t > 0$ and choose an input function ε_1 which satisfies the conditions of Lemma 4.3.5 and is continuous for $t > 0$. The first iterate ε_2 satisfies $\varepsilon_2(t) < \varepsilon_1(t)$ from this lemma. Furthermore, this result applies for all earlier times greater than zero; we have $\varepsilon_2(\tau) < \varepsilon_1(\tau)$ for $0 < \tau \leq t$. We recall that $\varepsilon_Q(t) \geq \varepsilon_L(t) = \varepsilon_o H(t)$ for $t \geq 0$, with equality only at $t = 0$. Therefore, our starting function ε_1 satisfies the conditions of Lemma 4.3.7, with the result that $\varepsilon_L(\tau) < \varepsilon_2(\tau)$ for $0 < \tau \leq t$. Moreover, by Lemma 4.3.3, ε_2 is monotone and continuous for $t > 0$, since ε_1 is. By continuing the iteration process, Lemma 4.3.4 guarantees that the succeeding outputs are less than their associated inputs. We thus have a monotone sequence of monotone functions which are continuous for $t > 0$ and bounded below by ε_L:

$$\varepsilon_L(\tau) \leq \varepsilon_{n+1}(\tau) \leq \varepsilon_n(\tau) \leq \ldots \leq \varepsilon_2(\tau) \leq \varepsilon_1(\tau), \qquad 0 \leq \tau \leq t, \tag{4.4.1}$$

with equality only at $\tau = 0$ and only if $\varepsilon_1(0) = \varepsilon_o$ (i.e., when the starting function takes the value of the solution at $t = 0$). We similarly construct a lower bound sequence by starting with an input function which is continuous for $t > 0$ and which satisfies the conditions of Lemma 4.3.8, producing an output ε_2 which is larger than ε_1 and, by Lemma 4.3.6, which is bounded above by ε_Q. The result is a monotone sequence of monotone functions which are continuous for $t > 0$ and bounded above by ε_Q:

$$\varepsilon_1(\tau) \leq \varepsilon_2(\tau) \leq \ \ldots \leq \varepsilon_n(\tau) \leq \varepsilon_{n+1}(\tau) \leq \varepsilon_Q(\tau), \qquad 0 \leq \tau \leq t, \tag{4.4.2}$$

with equality only at $\tau = 0$ and only if $\varepsilon_1(0) = \varepsilon_o$.

From the discussion of section 4.2, we have that such upper or lower bound sequences converge to limit functions which may, in general, depend upon the starting functions. The following theorem guarantees that all such sequences generated from our iteration scheme necessarily converge to the same function for $U > U_o$.

Theorem 4.4.1. Existence and Uniqueness for $U > U_o$. *If $\varepsilon_1(t)$ is continuous for $t > 0$ and satisfies the conditions on ε_{in} in either Lemma 4.3.5 or Lemma 4.3.8, then the upper bound or lower bound sequence of functions produced by the iteration scheme (4.3.3) converges to a monotone function which is the unique solution to (4.3.1) among the class of functions which are continuous except for a finite jump discontinuity at $t = 0$. Furthermore, these sequences converge uniformly on the interval $[0, t]$ for any finite $t > 0$.*

Proof: We first remark that there exist suitable starting functions, ε_1, examples of which are ε_L and ε_Q for lower and upper bound sequences, respectively. We prove that an upper bound sequence converges to a solution. The same arguments apply for lower bound sequences. We denote by ε^* a function to which any upper bound sequence (4.4.1) converges. From (4.3.3) we have:

$$\varepsilon^*(t) = \lim_{n\to\infty} \varepsilon_{n+1}(t) = \lim_{n\to\infty} F\Big(U^2(J' \star \varepsilon_n)(t)\Big)$$
$$= F\Big(U^2 \lim_{n\to\infty}(J' \star \varepsilon_n)(t)\Big),$$

since F is continuous. Then by dominated convergence [4.2],

$$\varepsilon^*(t) = F\Big(U^2(J' \star \varepsilon^*)(t)\Big),$$

since the integrands are dominated by the first one in the sequence, and this first one was chosen to be integrable. (For lower bound sequences, Lemma 4.3.6 provides the bound.) Thus ε^* is a solution. This limit function is monotone since it is the limit of a convergent sequence of monotone functions [4.2]. Now, a monotone function can be discontinuous only if it has finite jump discontinuities [4.2]. Since ε^* is a solution, it vanishes for $t < 0$ and can have only the single jump discontinuity which occurs at $t = 0$ (Property 5, §§3.3). Therefore, the monotonicity of ε^* implies its continuity for $t > 0$.

We now prove that the solution to (4.3.1) is unique among functions which are continuous except for a finite jump discontinuity at $t = 0$. We will show that an assumption of two different solutions leads to a contradiction. We begin by considering the difference $|\varepsilon_1 - \varepsilon_2|(t)$ of any two solutions, identical or not, at a time t which is larger than the greatest time, say t_o, before which the solutions agree (all solutions at these wavespeeds agree until at least $t = 0^+$):

$$|\varepsilon_1 - \varepsilon_2|(t) = \Big| F\Big(U^2(J' \star \varepsilon_1)(t)\Big) - F\Big(U^2(J' \star \varepsilon_2)(t)\Big) \Big|.$$

Using equation (2.3.16) to bound the right side, we obtain:

$$|\varepsilon_1 - \varepsilon_2|(t) \leq M\Big(U^2(J' \star \varepsilon_\beta)(t)\Big)U^2\Big|(J' \star (\varepsilon_1 - \varepsilon_2))(t)\Big|; \qquad \beta \in \{1, 2\},$$

with equality if and only if the convolutions are equal at t. Such an equality requires either that $\varepsilon_1(\tau)$ is identical to $\varepsilon_2(\tau)$ for $\tau \leq t$, or that the difference $(\varepsilon_1 - \varepsilon_2)(\tau)$ is not of one sign, where it is non-zero, for $\tau \leq t$, in view of Lemma 4.3.2. With the use of a triangle inequality, we pass the absolute value inside the integral to get:

$$|\varepsilon_1 - \varepsilon_2|(t) \leq M\left(U^2(J' \star \varepsilon_\beta)(t)\right)U^2(J' \star |\varepsilon_1 - \varepsilon_2|)(t),$$

with equality, now, if and only if $\varepsilon_1(\tau)$ is identical to $\varepsilon_2(\tau)$ for $\tau \leq t$.

We now assume that ε_1 differs from ε_2 somewhere on $[t_o, t]$. The continuity of ε_1 and ε_2 ensures that the difference $|\varepsilon_1 - \varepsilon_2|$ achieves its maximum somewhere on $[t_o, t]$. By hypothesis, $\varepsilon_1(t_o) = \varepsilon_2(t_o)$; hence, $|\varepsilon_1 - \varepsilon_2|$ is not equal to its maximum on the entire interval, and we have the strict inequality:

$$|\varepsilon_1 - \varepsilon_2|(t) < M\left(U^2(J' \star \varepsilon_\beta)(t)\right)U^2 J(t - t_o) \max_{\tau \in [t_o, t]} |\varepsilon_1 - \varepsilon_2|(\tau). \tag{4.4.3}$$

In obtaining (4.4.3), we have also used the monotonicity of J and the fact that $\varepsilon_1 - \varepsilon_2$ vanishes for $t \leq t_o$. Now, ε_β is a solution by hypothesis. We have from equations (3.1.10) and (2.3.13):

$$M(U^2 J' \star \varepsilon_\beta) = M(f(\varepsilon_\beta)) = \frac{1}{m(\varepsilon_\beta)} < \frac{1}{U^2 J_o}. \tag{4.4.4}$$

The inequality in (4.4.4) is obtained for increasing m since $\varepsilon_\beta(t) > \varepsilon_o$ for $t > 0$ and $m(\varepsilon_o) = U^2 J_o$. Thus,

$$M\left(U^2(J' \star \varepsilon_\beta)(t)\right)U^2 J_o < 1 \qquad \text{for all} \qquad t > 0.$$

Since $J(t)$ is continuous for $t > 0$, there is a time $t_1 > t_o$ such that

$$M\left(U^2(J' \star \varepsilon_\beta)(t_1)\right)U^2 J(t_1 - t_o) < 1. \tag{4.4.5}$$

Using this in equation (4.4.3), we have

$$|\varepsilon_1 - \varepsilon_2|(t_1) < \max_{\tau \in [t_o, t_1]} |\varepsilon_1 - \varepsilon_2|(\tau). \tag{4.4.6}$$

Hence, the maximum occurs at a time $t_1' < t_1$. Equation (4.4.5) is also satisfied at t_1' and we have:

$$|\varepsilon_1 - \varepsilon_2|(t_1') < |\varepsilon_1 - \varepsilon_2|(t_1').$$

This contradiction follows directly from the earlier assumption that $\varepsilon_1 \neq \varepsilon_2$ somewhere on $[t_o, t]$. We conclude that the solution is unique in the class of functions considered.

Lastly, the uniform convergence on an interval $[0, t]$ of the sequence of iteration functions follows from the continuity of ε^* and each member of $\{\varepsilon_n\}_{n=1}^{\infty}$, according to the following version [4.2] of

Dini's Theorem. *Let g, f_1, $f_2, \ldots$ be continuous functions on $[a, b]$ such that $f_1 \geq f_2 \geq \ldots$ and $f_n(t) \to g(t)$ as $n \to \infty$ for all $t \in [a, b]$. Then $f_n \to g$ uniformly.*

We note that Dini's theorem also applies for lower bound sequences $\{f_n\}_{n=1}^{\infty}$ by considering the upper bound sequence whose members are $g_n = g - f_n$ converging to zero in the statement of the theorem above. This concludes the proof of Theorem 4.4.1. □

4.5 $U_e < U < U_o$.

Recall in Section 3.3, we chose as a candidate solution for these values of wavespeed a function which, for $t \to \infty$, is asymptotic to an exponential solution of the linear problem in which $f(\varepsilon) \equiv \varepsilon$. With this in mind and, since solutions for these values of U have no finite jump discontinuities (§§3.3), we consider continuous solutions of the form:

$$\varepsilon(t) = w(t)e^{rt}, \tag{4.5.1}$$

where w is continuous and satisfies

$$\lim_{t \to -\infty} w(t) = 1, \tag{4.5.2}$$

and r is the unique solution to (3.2.3). For simplicity, and without loss of generality, we have set the time shift t_c in equation (3.2.2) equal to zero. We again prove that upper and lower bound sequences from the iteration scheme (4.3.3) converge to a solution of (4.3.1). We will also prove that the solution is unique among functions which have the form of (4.5.1) for these wavespeeds.

We begin with the proofs of some preparatory lemmas, the first of which shows that e^{rt} (which solves the linear problem) does not solve the nonlinear problem. Rather, it starts a decreasing (upper bound) sequence when $\varepsilon_1 = e^{rt}$ in the iteration scheme. Additionally, it is an upper bound for increasing sequences when $\varepsilon_1 < e^{rt}$.

Lemma 4.5.1. *If $\varepsilon_{\text{in}}(\tau) \leq e^{r\tau}$ for all $\tau \leq t$, where r satisfies (3.2.3), then $\varepsilon_{\text{out}}(t) < e^{rt}$.*

Proof: From Lemma 4.3.2 we have

$$J' * \varepsilon_{\text{in}} \leq J' * e^{rt},$$

with equality if and only if ε_{in} is identical to e^{rt}. Now,

$$U^2 J' * e^{rt} = U^2 r \overline{J}(r) e^{rt} = e^{rt},$$

since $U^2 r \overline{J}(r) = 1$. Therefore,

$$\varepsilon_{\text{out}}(t) = F\left(U^2 (J' * \varepsilon_{\text{in}})(t)\right) < F\left(e^{rt}\right) < e^{rt},$$

where we have used $F(f) < f$ to obtain the last inequality. □

To obtain a lower bound for upper bound sequences (and at the same time a function which starts a lower bound sequence), we use the lower bound on the function F given in equation (2.3.19) and repeated here for convenience:

$$F(f) > f - kf^{\gamma}, \qquad k > 0, \qquad \gamma > 1. \tag{4.5.3}$$

By using this bound when $\varepsilon_{\text{in}} = e^{rt}$ in (4.3.2), we find that

$$\varepsilon_{\text{out}}(t) > e^{rt} - kU^2\gamma r\overline{J}(\gamma r)e^{\gamma rt}.$$

We note that

$$U^2\gamma r\overline{J}(\gamma r) < 1 \qquad \text{for} \qquad \gamma > 1, \tag{4.5.4}$$

when $U^2 r\overline{J}(r) = 1$ because $r\overline{J}(r)$ is completely monotone (cf. equation (2.5.5) and §§3.2). To obtain a lower bound for more general inputs, we consider input functions at least as large as ε_b which we define as:

$$\varepsilon_b(t) = e^{rt} - Ce^{\gamma rt}, \qquad \gamma > 1, \tag{4.5.5}$$

for some positive constant C. If we choose C to be the positive constant given by

$$C = \frac{k}{1 - U^2\gamma r\overline{J}(\gamma r)}, \tag{4.5.6}$$

then ε_b is the desired lower bound, at least on some interval $(-\infty, t)$, according to

Lemma 4.5.2. *If $\varepsilon_{\text{in}}(\tau) \geq \varepsilon_b(\tau)$ for all $\tau \leq t$, where r satisfies (3.2.3) and C is given in (4.5.6), then there exists a finite t_b such that $\varepsilon_{\text{out}}(t) > \varepsilon_b(t)$ for all $t < t_b$.*

Proof:

$$\begin{aligned}
\varepsilon_{\text{out}}(t) &\geq F\Big(U^2(J' \star \varepsilon_b)(t)\Big) \\
&= F\Big(e^{rt} - CU^2\gamma r\overline{J}(\gamma r)e^{\gamma rt}\Big) \\
&> e^{rt} - CU^2\gamma r\overline{J}(\gamma r)e^{\gamma rt} - ke^{\gamma rt}[1 - CU^2\gamma r\overline{J}(\gamma r)e^{(\gamma-1)rt}]^{\gamma},
\end{aligned}$$

where we have used (4.5.3) to bound F and have factored $e^{\gamma rt}$ out of the bracketed quantity. This remaining bracketed quantity raised to any power $\gamma > 1$ is a well-defined positive number smaller than unity for $t < t_b$, where

$$t_b = -\frac{\ln(CU^2\gamma r\overline{J}(\gamma r))}{(\gamma - 1)r}. \tag{4.5.7}$$

Therefore,

$$\varepsilon_{\text{out}}(t) > e^{rt} - (CU^2\gamma r\overline{J}(\gamma r) + k)e^{\gamma rt}, \qquad t < t_b;$$

$$= \varepsilon_b(t), \qquad t < t_b;$$

by the definition of C. The time t_b is finite for the constants considered. □

We still do not have a lower bound for all t, but we can use ε_b to construct one. Observe that ε_b has a single maximum. We define ε_L to be the continuous function equal to ε_b until its maximum and equal to this maximum thereafter:

$$\varepsilon_L(t) = \begin{cases} \varepsilon_b(t), & t < t_m; \\ \varepsilon_b(t_m), & t \geq t_m; \end{cases} \tag{4.5.8}$$

where

$$t_m = -\frac{\ln(\gamma C)}{(\gamma - 1)r}. \tag{4.5.9}$$

We note that $t_m < t_b$, since equation (4.5.4) and $\gamma > 1$ imply that the argument of the logarithm in (4.5.9) is larger than that in (4.5.7). We now assert that ε_L is the desired lower bound for upper bound sequences and that it starts a lower bound sequence upon iteration.

Lemma 4.5.3. *If $\varepsilon_{\text{in}}(\tau) \geq \varepsilon_L(\tau)$ for all $\tau \leq t$ and ε_L defined in (4.5.8), then $\varepsilon_{\text{out}}(t) > \varepsilon_L(t)$.*

Proof: If $\varepsilon_{\text{in}} \equiv \varepsilon_L$, then $\varepsilon_{\text{out}}(t) > \varepsilon_L(t)$ for all $t < t_m$ by Lemma 4.5.2. Furthermore, ε_{out} is monotone and continuous according to Lemma 4.3.3, since ε_L is both monotone and continuous by construction. Therefore, $\varepsilon_{\text{out}}(t) > \varepsilon_L(t)$ for all $t \geq t_m$ If ε_{in} is not identical to ε_L, the result is immediate from what we have just proven and Lemma 4.3.4. □

The foregoing lemmas of this section provide starting functions and bounds for iteration, all of which have the desired properties (4.5.1) and (4.5.2). We now have all we need to prove the main theorem for these wavespeeds.

Theorem 4.5.1. Existence and Uniqueness for $U_e < U < U_o$. *If, for all real t, $\varepsilon_1(t) = e^{rt}$ where r satisfies (3.2.3), or if $\varepsilon_1(t) = \varepsilon_L(t)$ given in (4.5.8), then the iteration scheme (4.3.3) produces an upper bound or a lower bound sequence, respectively, which converges to a continuous monotone function on $(-\infty, +\infty)$. In either case, the limit function is the unique solution to (4.3.1) among the class of continuous functions which are asymptotic to e^{rt} for $t \to -\infty$. Moreover, these sequences converge uniformly on the interval $(-\infty, t]$ for any finite t.*

Proof: If $\varepsilon_1 = e^{rt}$ ($\varepsilon_1 = \varepsilon_L$) the use of Lemma 4.5.1 (Lemma 4.5.3) followed by repeated application of Lemma 4.3.4 shows that the iteration scheme produces a decreasing (increasing) sequence of functions bounded below by ε_L (above by e^{rt}) according to Lemma 4.5.3 (Lemma 4.5.1). Each member of the sequence is monotone and continuous according to Lemma 4.3.3, since ε_1 is monotone and continuous. Thus, the sequence converges to a monotone function ε^* on $(-\infty, +\infty)$. This limit function is a solution to (4.3.1) by dominated convergence, as in the proof of Theorem 4.4.1. Since a solution at these wavespeeds cannot have any finite jump discontinuities and since a monotone function can be discontinuous only if it has such discontinuities, we conclude that ε^* is continuous. Furthermore, ε^* is asymptotic to e^{rt} as $t \to -\infty$ since

$$e^{rt} - Ce^{\gamma rt} < \varepsilon^*(t) < e^{rt}, \qquad t < t_b,$$

where C is given in (4.5.6) and t_b in (4.5.7). Therefore,

$$\lim_{t \to -\infty} \frac{\varepsilon^*(t)}{e^{rt}} = 1.$$

To prove that ε^* is the unique continuous function having this asymptotic property, we write ε^* in the form given in (4.5.1) and (4.5.2) to obtain a bound on any such convolution $U^2 J' \star \varepsilon^*$:

$$\begin{aligned} U^2(J' \star \varepsilon^*)(t) &= U^2 \int_0^\infty \varepsilon^*(t-\tau)\, dJ(\tau) \\ &= e^{rt} U^2 \int_0^\infty w^*(t-\tau) e^{-r\tau}\, dJ(\tau) \\ &\leq e^{rt} \max_{\tau \leq t} |w^*(\tau)| U^2 \int_0^\infty e^{-r\tau}\, dJ(\tau) \\ &= e^{rt} \max_{\tau \leq t} |w^*(\tau)|, \end{aligned} \tag{4.5.10}$$

since the last integral is $r\overline{J}(r)$ and $U^2 r\overline{J}(r) = 1$ by the choice of r. Now, we assume that there are two different solutions ε_1 and ε_2 and obtain a contradiction. We consider the difference $|\varepsilon_1 - \varepsilon_2|(t)$ as in the proof of Theorem 4.4.1; using the bound in (4.5.10) we obtain:

$$|\varepsilon_1 - \varepsilon_2|(t) < e^{rt} M\left(U^2(J' \star \varepsilon_\beta)(t)\right) \max_{\tau \leq t} |w_1 - w_2|(\tau), \qquad \beta \in \{1, 2\},$$

with strict inequality since w_1 and w_2 agree at $-\infty$. When we divide by e^{rt} and use the fact that M decreases from $M(0) = 1$, we find that $w_1 - w_2$ must satisfy

$$|w_1 - w_2|(t) < \max_{\tau \leq t} |w_1 - w_2|(\tau) \qquad \text{for all } t.$$

From arguments identical to those following equation (4.4.6) in the proof of Theorem 4.4.1, we find that the assumption of two different continuous solutions leads directly to a contradiction.

To prove that the sequence $\{\varepsilon_n\}_{n=1}^\infty$ converges uniformly to ε^* on any interval $(-\infty, t]$, we note that $|\varepsilon_n - \varepsilon^*|(\tau) < e^{r\tau}$ since $\varepsilon_n(\tau)$ and $\varepsilon^*(\tau)$ are both bounded above by $e^{r\tau}$ and below by zero. For any $\delta > 0$ and for all $n > 1$ we have $|\varepsilon_n - \varepsilon^*|(\tau) < \delta$ whenever $\tau \leq t_\delta$, where $t_\delta = (\ln \delta)/r$. Thus, the convergence is uniform on $(-\infty, t_\delta]$. If $t > t_\delta$ the uniform convergence on $[t_\delta, t]$ afforded by Dini's Theorem (§§4.4) extends the result to $(-\infty, t]$. □

4.6 $U = U_o$.

For this particular value of wavespeed, our candidate solutions are continuous and vanish for $t < 0$. As before, our plan is to construct upper and lower bound sequences of such functions and to prove that they converge to a solution of equation (4.3.1). We will prove that this problem has a unique solution among the class of continuous functions which vanish for $t < 0$.

We begin by deriving an equivalent form of (4.3.1) useful at this value of wavespeed. We add and subtract $J_o H(t-\tau)$ inside the convolution integral in equation (3.1.10) and evaluate the added term to obtain:

$$f(\varepsilon) = U^2 J_o \varepsilon + U^2 (J - J_o)' * \varepsilon. \tag{4.6.1}$$

Using $U^2 J_o = 1$, we write this in terms of the *reduced strain curve* $f(\varepsilon) - \varepsilon$:

$$f(\varepsilon) - \varepsilon = \chi' * \varepsilon, \tag{4.6.2}$$

where

$$\chi(t) = \frac{J(t) - J_o}{J_o} H(t) \tag{4.6.3}$$

is the *normalized compliance* which vanishes for $t < 0$; χ is identically zero if and only if the material is purely elastic. We note that the reduced strain curve is monotone and, therefore, has an inverse, since $f(\varepsilon) - \varepsilon = (m(\varepsilon) - 1)\varepsilon$, each factor of which is strictly increasing in ε. We assume, as in equation (2.3.18), that there are constants $k > 0$ and $\gamma > 1$ such that

$$f(\varepsilon) - \varepsilon \sim k\varepsilon^{\gamma}, \qquad \varepsilon \downarrow 0. \tag{4.6.4}$$

In terms of the *scaled strain* $u(t)$ defined by

$$u(t) = k^{\frac{1}{\gamma - 1}} \varepsilon(t), \tag{4.6.5}$$

we consider the problem equivalent to (4.3.1) given by

$$u(t) = G\Big(\big(\chi' \star u\big)(t)\Big), \tag{4.6.6}$$

where G is **not** the modulus; rather, we now denote by G the strictly increasing inverse of the reduced strain curve after scaling:

$$G(g) = g^{1/\gamma} L(g), \qquad L(0) = 1, \tag{4.6.7}$$

in which L is an otherwise unspecified slowly-varying function. Solutions $u(t)$ to (4.6.6) with (4.6.4) and (4.6.5) are solutions to (4.3.1). The corresponding iteration scheme is:

$$u_{n+1}(t) = G\Big(\big(\chi' \star u_n\big)(t)\Big)\,; \qquad n = 1, 2, \ldots . \tag{4.6.8}$$

The quasi-elastic solution $u_Q = k^{\frac{1}{\gamma-1}} \varepsilon_Q$ satisfies

$$u_Q(t) = G\big(\chi(t) u_Q(t)\big). \tag{4.6.9}$$

We continue with our now familiar attack on the problem. The iteration scheme (4.6.8) is amenable to a set of statements analogous to Lemmas 4.3.3, 4.3.4. and 4.3.6 which allows us to assert, without a redundant proof, that u_Q starts a decreasing sequence of functions upon iteration and that it may be used as an upper bound for an increasing sequence.

Lemma 4.6.1. *If u_1 is monotone and continuous, vanishes for $t < 0$, and satisfies $u_1(\tau) \le u_Q(\tau)$ for all $\tau \le t$, then $u_2(t)$ is a continuous monotone function of t and, when the material is not purely elastic, u_2 satisfies $u_2(t) < u_Q(t)$, for $t > 0$*

We need only exhibit a function u_L which is a lower bound for a sequence started by u_Q and which starts an increasing sequence bounded above by u_Q. To do so, we use a lower bound on the normalized compliance χ. Since $\chi(0) = 0$ and its derivative is completely monotone (§§2.5), χ is convex down. For small enough times, it is bounded below by a line of slope a:

$$\chi(t) > at \qquad \text{for} \qquad 0 < t < t_a, \tag{4.6.10}$$

where

$$0 < a = \frac{\chi(t_a)}{t_a} < \chi'(0^+) \leq \infty. \tag{4.6.11}$$

We also use a lower bound on the function G given in (4.6.7). Since $L(g) \to 1$ as $g \downarrow 0$, there are positive constants δ and g_δ such that

$$G(g) > g^{1/\gamma}(1 - \delta) \qquad \text{for all} \qquad g \leq g_\delta. \tag{4.6.12}$$

Furthermore, we can always take $\delta < 1$ since G increases from $G(0) = 0$. Using these bounds on χ and G, we will show that a lower bound for decreasing sequences (and a starting function for an increasing sequence) is given by

$$u_L(t) = \begin{cases} r\, t^{\frac{1}{\gamma-1}}, & t < t_r; \\ r\, t_r^{\frac{1}{\gamma-1}}, & t \geq t_r; \end{cases} \tag{4.6.13}$$

for small enough values of the positive constants r and t_r.

Lemma 4.6.2. *There are positive constants* r *and* t_r *such that, if* u_1 *is a continuous monotone function which vanishes for* $t < 0$ *and which satisfies* $u_1(\tau) \geq u_L(\tau)$ *for all* $\tau \leq t$, *with* u_L *defined in equation (4.6.13), then* $u_2(t)$ *is a continuous monotone function of* t *when the material is not purely elastic, and* $u_2(t) > u_L(t)$ *for all* $t > 0$. *Furthermore,* $u_Q(t) > u_L(t)$ *for all* $t > 0$.

Proof: Fix $\delta_1 \in (0, 1)$ and let

$$r_1 = \left(a\,\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma-1}} (1 - \delta_1)^{\frac{\gamma}{\gamma-1}}.$$

We choose a value of $r \leq r_1$ and define

$$u_r(t) = r\, t^{\frac{1}{\gamma-1}}. \tag{4.6.14}$$

Using the bound on χ given in (4.6.10), we have

$$(\chi' * u_r)(t) > ar\,\frac{\gamma-1}{\gamma}\, t^{\frac{\gamma}{\gamma-1}}, \qquad 0 < t \leq t_a.$$

From the lower bound on G in (4.6.12), there is a time $t_1 > 0$ such that

$$\begin{aligned} G\Big(\big(\chi' * u_r\big)(t)\Big) &> \left(ar\,\frac{\gamma-1}{\gamma}\right)^{1/\gamma} t^{\frac{1}{\gamma-1}}\,(1-\delta_1), \qquad 0 < t \leq \min\{t_1, t_a\} \\ &\geq u_r(t), \end{aligned}$$

since $r \leq r_1$.

Starting anew, we fix $\delta_Q \in (0,1)$ and define

$$r_Q = a^{\frac{1}{\gamma-1}}(1-\delta_Q)^{\frac{\gamma}{\gamma-1}}.$$

Now let u_r be as in (4.6.14) with some value of $r \leq r_Q$. Recall that $u_Q = G(\chi u_Q)$. Using the lower bound on χ and the monotonicity of G, we have

$$u_Q(t) > G\big(at\,u_Q(t)\big), \qquad 0 < t < t_a.$$

The lower bound for G ensures there is a $t_Q > 0$ so that

$$u_Q(t) > \big(at\,u_Q(t)\big)^{1/\gamma}(1-\delta_Q), \qquad 0 < t \leq \min\{t_Q, t_a\}.$$

Solving for u_Q,

$$\begin{aligned} u_Q(t) &> (at)^{\frac{1}{\gamma-1}}(1-\delta_Q)^{\frac{\gamma}{\gamma-1}} \\ &\geq u_r(t), \end{aligned}$$

since $r \leq r_Q$.

With the above results, we have conditions which allow us to ensure that $u_Q > u_r$ and $G(\chi' * u_r) > u_r$. We now pick values of r and t_r which satisfy

$$r \leq \min\{r_1, r_Q\}$$

$$t_r \leq \min\{t_1, t_Q, t_a\}$$

and define $u_L(t)$ according to (4.6.13). If u_1 is identical to u_L we have just proven that $u_2 = G(\chi' \star u_1) > u_1$ for $t \in (0, t_r]$. Since u_L is monotone and continuous by construction, u_2 is, too, and the result is obtained for all $t > 0$. For functions u_1 not identical to u_L, the result follows from what we have just proven and a statement for this iteration scheme analogous to Lemma 4.3.4. Similarly, for these values of r and t_r we have $u_Q > u_L$ for $t \in (0, t_r]$. Since u_Q is continuously increasing, the result holds for all $t > 0$. □

We are now prepared to state and prove the main theorem for this wavespeed.

Theorem 4.6.1. Existence and Uniqueness for $U = U_o$. *If $u_1(t)$ satisfies the hypotheses of either Lemma 4.6.1 or Lemma 4.6.2, then the iteration scheme (4.6.8) produces an upper bound or lower bound sequence, respectively, which converges to a monotone function satisfying equation (4.6.6) for the scaled strain u. The corresponding strain ε from equation (4.6.5) is the unique solution to (4.3.1) among the class of continuous functions which vanish for $t < 0$. Furthermore, these sequences converge uniformly on the interval $[0, t]$ for any finite $t > 0$.*

Proof: The use of Lemma 4.6.1 (Lemma 4.6.2) followed by repeated application of a statement for the iteration scheme (4.6.8) analogous to Lemma 4.3.4 produces a decreasing (increasing) sequence of functions, each member of which is monotone and continuous. The sequence is bounded below by u_L (above by u_Q) and, therefore, converges to a monotone function u^* on $(-\infty, +\infty)$. By dominated convergence, u^* is a solution to (4.6.6); the corresponding strain ε^* satisfies equation (4.3.1) and is, therefore, continuous according to the arguments in Theorem 4.4.1.

Uniqueness of the solution is obtained from the original formulation (4.3.1) in terms of F; the proof is identical to that given in Theorem 4.4.1. In this case, we note that equation (4.4.4) holds, since $\varepsilon_\beta(t) > \varepsilon_o = 0$ for $t > 0$ and since $m(0) = 1 = U^2 J_o$.

Lastly, the uniform convergence of the sequence of functions on an interval $[0, t]$ follows from Dini's Theorem, as in the proof of Theorem 4.4.1. □

4.7 A theorem for bounds via trial and error.

In the previous sections, we have exhibited means for constructing solutions from sequences of functions which globally bound the solution, either from above or below. The first members of these sequences provide the loosest bounds. Moreover, the lower bounds we used, ε_L and u_L, do not resemble the global behaviour of the solutions. For the purpose of applications, bounds for the solution up to a time T can be useful if they can be made tight to some degree without requiring many iterations in the constructive iteration scheme. The following theorem asserts the usefulness of a trial and error approach to finding starting functions which provide tighter bounds on the solution upon iteration. It allows us to make a guess for the starting function on an interval $(-\infty, T_1]$ and test its iteration. If the iterate moves up (down) on the sub-interval $(-\infty, T]$, for $T \leq T_1$, then the iterate is a lower bound (an upper bound) on the solution until T. In what follows, we denote by $x(t)$ a generic solution, either the strain $\varepsilon(t)$ or the scaled strain $u(t)$ and we let $\Phi(x_n)$ represent either $F(U^2 J' \star \varepsilon_n)$ or $G(\chi' \star u_n)$ as appropriate for the wavespeed under consideration.

Theorem 4.7.1. *Suppose x_U (x_L) is a known upper bound (lower bound) for a solution to (4.3.1) and for which $\Phi(x_U) < x_U$ ($\Phi(x_L) > x_L$). Choose a non-negative increasing function $x_1(t)$ which satisfies $x_1(t) \leq x_U(t)$ ($x_1(t) \geq x_L(t)$) for $t \in (-\infty, T_1]$ and which vanishes for $t < 0$ if $U \geq U_o$, or which has the form of (3.2.3) if $U_e < U < U_o$. If there is a time $T \leq T_1$ so that one iteration produces*

$\Phi(x_1(t)) = x_2(t) \geq x_1(t)$ *($x_2 \leq x_1$) for all $t \leq T$, then the chosen function x_1 starts a lower bound (an upper bound) sequence which converges to a solution on $(-\infty, T]$.*

Proof: Consider the case for an increasing (lower bound) sequence. The first iterate, x_2, is bounded above by x_U:

$$x_1(t) \leq x_2(t) = \Phi(x_1(t))$$
$$\leq \Phi(x_U(t)) < x_U(t), \qquad t \leq T,$$

since $x_1 \leq x_2$ and $x_1 \leq x_U$ for $t \leq T \leq T_1$, by hypothesis. Repeated application of a statement analogous to Lemma 4.3.4 produces the increasing sequence:

$$x_1(t) \leq x_2(t) < x_3(t) < \ldots < x_U(t), \qquad t \leq T,$$

at times where the functions are not identically zero. The same argument applies with reversed inequalities for decreasing (upper bound) sequences bounded below by x_L. Convergence to a solution with the appropriate properties is proven from the arguments in the proofs of Theorems 4.4.1, 4.5.1 and 4.6.1. □

CHAPTER 5: STEADY SHOCK AND ACCELERATION WAVES

5.1 Problem Statement and Results.

We first consider, in section 5.2, the explicit dependence of shock and acceleration wave solutions on the parametric wavespeed U. We show that $\varepsilon(t;U)$ is continuous in U for solutions which vanish for $t < 0$. In particular, the acceleration wave is the limit of shock waves as $U \downarrow U_o$. Along the way, we show for two shock or acceleration waves having different wavespeeds that the faster wave is everywhere larger than the slower wave, after the initial jump. We conclude with a theorem which is based on this result; it allows us to construct a solution for a wavespeed U_b *from* a known solution at a different wavespeed U_a, when both wavespeeds are at least as large as the acceleration wavespeed U_o.

In the next three sections, we consider the behaviour for $t \downarrow 0$ and $t \to \infty$ of solutions $\varepsilon(t)$ for shock and acceleration waves; i.e., for values of wavespeed $U \geq U_o$. We first obtain the small-time asymptotic forms of $\varepsilon(t) - \varepsilon_o$, where ε_o is the non-negative value the solution takes at $t = 0^+$. When the strain curve f satisfies the assumptions of section 2.3, the result obtained in section 5.3 for almost-all shock waves is:

$$\varepsilon(t) - \varepsilon_o \sim \alpha\, \chi(t), \qquad t \downarrow 0, \tag{5.1.1}$$

where

$$\alpha = \frac{f(\varepsilon_o)}{f'(\varepsilon_o) - (U/U_o)^2}, \tag{5.1.2}$$

and $\chi(t)$ is the normalized compliance introduced in equation (4.6.3). Moreover, the quasi-elastic shock solution has the same asymptotic form

Our small-time result for an acceleration wave, for which $\varepsilon_- = 0$, relies upon the additional assumption that $\chi(t)$ is regularly-varying for $t \downarrow 0$ with power $P(\chi) = p \in (0, 1]$. We find that $\varepsilon(t)$ is also regularly-varying. Its power is $P(\varepsilon) = q \in (0, \infty)$, where

$$q = \frac{p}{\gamma - 1}, \qquad \gamma > 1, \tag{5.1.3}$$

in which γ is the dominant nonlinear power in the reduced strain curve $f(\varepsilon) - \varepsilon$ for small values of ε, as in equation (4.6.4). In terms of the scaled strain, $u(t)$, defined in equation (4.6.5), the result we obtain in section 5.4 for acceleration waves is

$$u(t) \sim \left(R\,\chi(t)\right)^{\frac{1}{\gamma - 1}}, \qquad t \downarrow 0, \tag{5.1.4}$$

where

$$R = \frac{p!\,q!}{(p + q)!}. \tag{5.1.5}$$

Here the quasi-elastic solution differs. The result is

$$u_Q(t) \sim \left(\chi(t)\right)^{\frac{1}{\gamma - 1}}, \qquad t \downarrow 0. \tag{5.1.6}$$

The final section, 5.5, is devoted to examples of shock and acceleration waves constructed for specific materials from a numerical approximation to the iteration scheme.

5.2 Dependence of solutions on the wavespeed U.

We consider the explicit dependence of solutions ε on the parametric wavespeed U and we write $\varepsilon(t, U)$. We sometimes use the notation $\varepsilon_a(t) = \varepsilon(t, U_a)$.

We have shown in Properties 1 and 5 of section 3.3 that the equilibrium value and, for shock and acceleration waves, the initial jump value are strictly increasing functions of the wavespeed. It is natural to ask if such a statement is true for shock and acceleration waves for all times where the solution is non-zero. Indeed, it is, according to

Lemma 5.2.1. *Among solutions which vanish identically for $t < 0$, if $U_a < U_b$ then $\varepsilon(t, U_a) < \varepsilon(t, U_b)$ for all $t > 0$.*

Proof. Solutions which vanish for $t < 0$ are either shock or acceleration waves and correspond to wavespeeds $U \geq U_o$. For such solutions, the difference $\varepsilon_a(t) - \varepsilon_b(t)$ is continuous for $t > 0$, according to Theorems 4.4.1 and 4.6.1. Since $\varepsilon_a(0) < \varepsilon_b(0)$ for $U_a < U_b$ (§§3.3), the difference $\varepsilon_a(t) - \varepsilon_b(t)$ is either negative for all $t \geq 0$, and the proof is complete, or it is zero at some finite time $t_o > 0$. We will show that the latter case is contradictory. We subtract the equations satisfied by ε_a and ε_b

$$\begin{aligned} \varepsilon_a(t) - \varepsilon_b(t) &= F\left(U_a^2(J' \star \varepsilon_a)(t)\right) - F\left(U_b^2(J' \star \varepsilon_b)(t)\right) \\ &< F\left(U_b^2(J' \star \varepsilon_a)(t)\right) - F\left(U_b^2(J' \star \varepsilon_b)(t)\right), \end{aligned}$$

since F is strictly increasing and $U_a < U_b$, by hypothesis. Using equation (2.3.16) on the right side, we obtain:

$$\varepsilon_a(t) - \varepsilon_b(t) < M\left(U_b^2(J' \star \varepsilon_a)(t)\right) U_b^2 \left(J' \star (\varepsilon_a - \varepsilon_b)\right)(t).$$

Now, assume that t_o is the first time for which $\varepsilon_a - \varepsilon_b$ is zero. Since $\varepsilon_a(\tau) - \varepsilon_b(\tau) < 0$ for all $0 \leq \tau < t_o$, the convolution on the right side is negative. Furthermore, M is positive and we have at t_0

$$0 < -\left|M\left(U_k^2(J' * \epsilon_k)(t_0)\right) U_l^2 (J' * (\epsilon_k - \epsilon_l))(t_0)\right.$$

which contradicts the assumption that $\epsilon_k(t_0) - \epsilon_l(t_0) = \alpha$.

We now use the results of the preceding lemma to prove that $\epsilon(t;U)$ is continuous in U under restricted circumstances.

Theorem 5.2.1. *Among solutions which vanish identically for $t < 0$, $\epsilon(t;U)$ is a continuous function of U. In particular, the acceleration wave at $U = U_0$ is the limit of shock waves as $U \downarrow U_0$.*

Proof. We are again concerned with wavespeeds $U > U_0$. For $U > U_0$, continuity follows from showing that $\epsilon(t;U)$ is both right-continuous and left-continuous at U. We provide the details of the proof for right-continuity and indicate parenthetically a modification for the similar proof of left-continuity.

For any $U > U_0$, we choose a convergent sequence of wavespeeds $\{U_n\}_{n=1}^{\infty}$ for which $U_n \downarrow U$. For each n, Theorem 4.4.1 guarantees the existence of a shock solution

$$\epsilon(t;U_n) = F\left(U_n^2(J' * \epsilon(\cdot;U_n))(t)\right), \qquad n = 1, 2, \ldots \tag{5.2.1}$$

From Lemma 5.2.1, these monotone solutions form a decreasing sequence of functions, each member of which is bounded below by $\epsilon(t;U)$. There is a monotone function $\bar{\epsilon}(t)$ such that $\epsilon(t;U_n) \downarrow \bar{\epsilon}(t) \geq \epsilon(t;U)$. We may pass to the limit as $U_n \downarrow U$ in equation (5.2.1), since F is continuous and the convolution integrands are dominated by $\epsilon(t;U_1)$. (For increasing sequences used in the proof of left-continuity as $U_n \uparrow U$, the integrands are dominated by the shock solution for any wavespeed greater than U.) In the limit, we have

$$\bar{\epsilon}(t) = F\left(U^2(J' * \bar{\epsilon})(t)\right)$$

Hence, $\bar{\epsilon}(t)$ is a shock solution and has its only finite discontinuity at $t = 0$, before which it vanishes identically. Since $\bar{\epsilon}$ is monotone, it is continuous for $t > 0$. From the uniqueness of such solutions provided by Theorem 4.4.1, we have $\bar{\epsilon}(t) = \epsilon(t, U)$ and we have proven that

$$\lim_{U_n \downarrow U} \epsilon(t, U_n) = \bar{\epsilon}(t) = \epsilon(t, U)$$

We similarly prove that there is a function $\underline{\epsilon}(t)$ such that for $U > U_o$

$$\lim_{U_n \uparrow U} \epsilon(t, U_n) = \underline{\epsilon}(t) = \epsilon(t, U)$$

Hence $\epsilon(t, U)$ is continuous in U.

To obtain the acceleration wave as the limit of shock waves for $U \downarrow U_o$, we follow the proof of right-continuity above with the following arguments. Since each monotone member of the sequence $\{\epsilon(t, U_n)\}_{n=1}^{\infty}$ vanishes for $t < 0$ and since the limit solution $\epsilon(t)$ can have no discontinuities at wavespeed U_o, we have that $\epsilon(t)$ is continuous and vanishes for $t < 0$. From Theorem 4.6.1, such a solution is unique. Hence

$$\lim_{U_n \downarrow U_o} \epsilon(t, U_n) = \epsilon(t) = \epsilon(t, U_o).$$

Lemma 5.2.1 spawns another result which is, perhaps, more useful than the theoretical result of continuity of solutions in wavespeed. It allows us to use the information in a known solution at a particular wavespeed to generate the solution at any other wavespeed, when both wavespeeds in question are at least as large as that of the acceleration wave.

Theorem 5.2.2. *Suppose $U_a, U_b > U_o$ are given, with $U_a \neq U_b$, and let $U = U_b$ in the iteration scheme (4.3.3). If the solution at wavespeed U_a is known to be $\epsilon(\tau, U_a) = \epsilon_a(\tau)$ for $\tau \in [0, t]$, then upon iteration, ϵ_a starts an upper bound (a* *lower bound) sequence of functions when $U_a > U_b$ ($U_a < U_b$). This sequence converges uniformly on $[0, t]$ to the shock or acceleration wave solution $\varepsilon_b(\tau) = \varepsilon(\tau; U_b)$.*

Proof. We prove the case $U_a > U_b$. The same arguments apply with the inequalities reversed for $U_a < U_b$. Suppose $U_a > U_b$ and let $\varepsilon_1 = \varepsilon_a$ in the iteration scheme. Then for $\tau \in [0, t]$

$$\varepsilon_2(\tau) = F\Big(U_b^2(J' \star \varepsilon_a)(\tau)\Big) < F\Big(U_a^2(J' \star \varepsilon_a)(\tau)\Big) = \varepsilon_a(\tau) = \varepsilon_1(\tau)$$

where we have used the increasing nature of F to obtain the inequality. Repeated iteration produces a decreasing sequence which is bounded below by $\varepsilon_b(\tau)$ according to Lemma 4.3.4 and the following inequality,

$$\varepsilon_2(\tau) = F\Big(U_b^2(J' \star \varepsilon_a)(\tau)\Big) > F\Big(U_b^2(J' \star \varepsilon_b)(\tau)\Big) = \varepsilon_b(\tau),$$

since $\varepsilon_a > \varepsilon_b$ according to Lemma 5.2.1 when $U_a > U_b$. The bounded decreasing sequence converges uniformly to the unique solution appropriate for the wave-speed $U_b > U_o$, according to either Theorem 4.4.1 or Theorem 4.6.1. □

5.3 Shocks as $t \downarrow 0$.

We begin by deriving an equation satisfied by $\varepsilon(t) - \varepsilon_o$ for $t > 0$. We subtract $f(\varepsilon_o) = U^2 J_o \varepsilon_o$ from each side of equation (3.1.10)

$$f(\varepsilon(t)) - f(\varepsilon_o) = U^2(J' \star \varepsilon)(t) - U^2 J_o \varepsilon_o \tag{5.3.1}$$

We then use (4.6.3) to write $J(t) = J_o\big[H(t) + \chi(t)\big]$ in the convolution integral in (5.3.1). We obtain

$$f(\varepsilon(t)) - f(\varepsilon_o) = \lambda(\varepsilon(t) - \varepsilon_o) + \lambda(\chi' \star \varepsilon)(t), \tag{5.3.2}$$

where

$$\chi = t^2 J \;\; (t/t)^2 \tag{5.3.3}$$

Since both ... and χ vanish for $t < 0$ we have

$$(\chi' * \cdot)(t) = \int_0^t \cdot(t - \tau)\, d\chi(\tau) \tag{5.3.4}$$

Let us define

$$\varphi(t) = \ldots(t) - \ldots H(t) \tag{5.3.5}$$

When we use ... $H(t)$ in the integral of (5.3.4) we obtain

$$(\chi' * \cdot)(t) = \ldots \chi(t) + (\chi' * \varphi)(t) \tag{5.3.6}$$

Using the previous two equations in equation (5.3.2) we obtain an equation for φ

$$f(\ldots + \varphi(t)) - f(\ldots) - \lambda \varphi(t) = f(\ldots)\chi(t) + \lambda(\chi' * \varphi)(t) \tag{5.3.7}$$

where we have used $f(\ldots) = \lambda \ldots$. Since f is assumed to have a piecewise continuous derivative (5.2.3) we may write

$$(f'(\ldots) - \lambda)\varphi(t) L(\varphi(t)) = f(\ldots)\chi(t) + \lambda(\chi' * \varphi)(t), \tag{5.3.8}$$

where L is a slowly-varying function for which $L(0) = 1$. Equation (5.3.8) will lead us to the result for almost all shock waves. The exceptions occur at wavespeeds for which $f'(\ldots) = \lambda$ or $f'(\ldots) = \lambda$ corresponding, for example, to points A and B in Figure 2.1. In practice, waves traveling at these exact values of wavespeed have no probability of being excited in a material, nor do these cases present any interesting mathematical phenomena.

We now show that the left side of equation (5.3.7), equivalently the left side of (5.3.8), is an increasing function of φ and, therefore, invertible. Its derivative with respect to φ is positive, according to the relations

$f'(\epsilon_0 + \gamma) > m(\epsilon_0 + \gamma) > m(\epsilon_0) = \lambda$, which follow from equations (2.3.6), (3.3.4), (5.3.3), and the monotonicity of m. Therefore, we may invert equation (5.3.8) for $\gamma(t)$ and write

$$\gamma(t) = \left[\alpha\chi(t) + \beta(\chi' * \gamma)(t)\right] L\left(\alpha\chi(t) + \beta(\chi' * \gamma)(t)\right) \tag{5.3.9}$$

where α is given in equation (5.1.2),

$$\beta = \frac{\lambda}{f'(\epsilon_0) - \lambda} \tag{5.3.10}$$

and L is a slowly-varying function for which $L(0) = 1$.

Now consider the limiting behaviour of equation (5.3.9) for $t \downarrow 0$. Both χ and the convolution $\chi' * \gamma$ vanish. Furthermore the convolution is the higher order term, for bounds like equation (2.6.9) provide that

$$0 < \frac{(\chi' * \gamma)(t)}{\chi(t)} < \gamma(t), \qquad t > 0. \tag{5.3.11}$$

Since χ and $\chi' * \gamma$ are continuous, and since γ vanishes as $t \downarrow 0$, (5.3.11) implies

$$\lim_{t \downarrow 0} \frac{(\chi' * \gamma)(t)}{\chi(t)} = 0. \tag{5.3.12}$$

Using this result in (5.3.9), we obtain

$$\lim_{t \downarrow 0} \frac{\gamma(t)}{\alpha\chi(t)} = 1, \tag{5.3.13}$$

which is the result stated in equation (5.1.1). The same result applies for the quasi-elastic solution obtained by replacing the convolution by multiplication in (5.3.9).

5.4 Acceleration Waves as $t \downarrow 0$.

We first obtain the asymptotic form of the quasi-elastic scaled strain $u_Q(t)$ from its governing equation, (4.6.9), using equation (4.6.7) for the function G. Simple algebra yields

$$u_Q(t) = (\chi(t))^{\frac{1}{\gamma - 1}} \left(L(u_Q(t)\chi(t))\right)^{\frac{1}{\gamma - 1}}. \tag{5.4.1}$$

Thus,

$$\lim_{t \downarrow 0} \frac{u_Q(t)}{(\chi(t))^{\frac{1}{\gamma-1}}} = 1 \tag{5.4.2}$$

as asserted in equation (5.1.6). If χ is regularly-varying and $P(\chi) = p \in (0, 1]$, then u_Q is also regularly varying. Its power $P(u_Q) = q \in [0, \infty)$ is given in equation (5.1.3).

To obtain the asymptotic expression for $u(t)$ given in (5.1.4), we use the iteration scheme (4.6.5) starting with $u_1 = u_Q$ to construct a convergent sequence of functions $\{u_n\}_{n=1}^{\infty}$ and convergent sequences of regularly-varying upper and lower bounds. Using equation (2.4.10) for the convolution of regularly-varying functions, we have for the convolution in the first iterate as $t \downarrow 0$

$$\begin{aligned} (\chi' * u_1)(t) &\sim R\,\chi(t) u_1(t) \\ &= R\,u_1^{\gamma}(t), \end{aligned} \tag{5.4.3}$$

where R is given in (5.1.5) and we have used (5.1.6). Equation (5.4.3) implies the existence of positive constants θ and t_θ such that

$$(1-\theta) R\,u_1^{\gamma}(t) < (\chi' * u_1)(t) < (1+\theta) R\,u_1^{\gamma}(t), \qquad 0 < t < t_\theta \tag{5.4.4}$$

Hereafter, we consider only the upper bound inequality in (5.4.4) for simplicity of presentation; the same arguments apply to the lower bound. From the iteration scheme $u_2 = G(\chi' * u_1)$ with G strictly increasing. Applying this and equation (1.6.7) to (5.4.4), we have

$$u_2(t) < \left[(1+\theta)R\right]^{1/\gamma} u_1(t)\, L\Big((\chi' * u_1)(t)\Big) \tag{5.4.5}$$

Since $L(0) = 1$, there are constants $\delta_n > 0$ for $n = 1, 2, \ldots$, such that

$$L\Big((\chi' * u_n)(t)\Big) < (1+\delta_n), \qquad 0 < t < t_\delta \tag{5.4.6}$$

Thus, for $0 < t < t_\theta$, the bound in (5.4.5) may be written

$$u_2(t) < \left[(1+\theta)R\right]^{1/\gamma}(1+\delta_1)\,u_1(t). \tag{5.4.7}$$

Repeated application of (5.4.3), (4.6.8), and (5.4.6) produces the sequence of bounding relations.

$$\begin{aligned} u_{n+1}(t) < \left((1+\theta)R\right)^{\frac{1}{\gamma}+\frac{1}{\gamma^2}+\cdots+\frac{1}{\gamma^n}} \\ \times\ (1+\delta_1)^{\frac{1}{\gamma^{n-1}}}(1+\delta_2)^{\frac{1}{\gamma^{n-2}}}\cdots(1+\delta_{n-1})^{\frac{1}{\gamma}} \\ \times\ u_1(t)L\Big((\chi'\star u_n)(t)\Big). \end{aligned} \tag{5.4.8}$$

The sequence $\{u_n(t)\}_{n=1}^{\infty}$ converges to a function $u(t)$, according to Theorem 4.6.1. Since L is continuous and since the functions u_n are dominated by u_Q, we have $L(\chi'\star u_n) \to L(\chi'\star u)$ as $n \to \infty$. As in (5.4.6), $L(\chi'\star u)$ is also bounded.

$$L\Big((\chi'\star u)(t)\Big) < 1+\delta_\infty, \qquad 0 < t < t_\theta \tag{5.4.9}$$

We have implicitly defined a convergent and, therefore, bounded sequence $\{\delta_n\}_{n=1}^{\infty}$. Let us define

$$\delta = \sup_n\{\delta_n\}. \tag{5.4.10}$$

Then,

$$(1+\delta_1)^{\frac{1}{\gamma^{n-1}}}(1+\delta_2)^{\frac{1}{\gamma^{n-2}}}\cdots(1+\delta_{n-1})^{\frac{1}{\gamma}}L(\chi'\star u_n) < (1+\delta)^{1+\frac{1}{\gamma}+\frac{1}{\gamma^2}+\cdots+\frac{1}{\gamma^n}} \tag{5.4.11}$$

Using (5.4.11) in (5.4.8), we pass to the limit as $n \to \infty$, and sum the geometric series in the exponents to obtain the upper bound:

$$u(t) < \left[(1+\theta)R\right]^{\frac{1}{\gamma-1}}(1+\delta)^{\frac{\gamma}{\gamma-1}}u_1(t), \qquad 0 < t < t_\theta \tag{5.4.12}$$

From similar arguments for lower bounds on the iterates, there is a positive constant $\delta' < 1$ for an analogous lower bound on $u(t)$ which, combined with equation (5.4.12) provides

$$(1-\delta')^{\frac{\gamma}{\gamma-1}}(1-\theta)^{\frac{1}{\gamma-1}} < \frac{u(t)}{R^{\frac{1}{\gamma-1}}u_1(t)} < (1+\delta)^{\frac{\gamma}{\gamma-1}}(1+\theta)^{\frac{1}{\gamma-1}}, \qquad 0 < t < t_\theta \tag{5.4.13}$$

As $t_\theta \downarrow 0$, the constants δ, δ', and θ can be taken arbitrarily small. Thus, the extreme members of (5.4.13) tend to unity and we have

$$u(t) \sim R^{\frac{-1}{\gamma-1}}\, u_1(t), \qquad t \downarrow 0 \tag{5.4.14}$$

Since we started with $u_1 = u_Q$, equations (5.1.6) and (5.4.14) imply the result stated in equation (5.1.4). From the assumption that χ is regularly-varying with power p, we have that u_Q, u, and, hence, ε are regularly-varying with power q.

We note that the asymptotic relations for u and u_Q given in equations (5.1.4) and (5.1.6), respectively, are **exact** when the normalized compliance has a *power-law* form, $\chi(t) = \chi_0 t^p$, with constants χ_0 and p, and the strain curve is given by $f(\varepsilon) = \varepsilon + k\varepsilon^\gamma$.

The *initial power of the acceleration wave*, $P(\varepsilon) \equiv q = p/(\gamma - 1)$, can be the same for different materials characterized by their nonlinearity and compliance parameters γ and p. The difference in such waves having the same initial power is embodied in the ratio of factorials, R. For materials which are strongly nonlinear at small strains, γ is close to unity and the power q is large. We use Stirling's approximation [5.1] to the factorials for large q and obtain the asymptotic form of R

$$\begin{aligned} R &= \frac{p!\,q!}{(p+q)!} \\ &\sim \frac{p!\,e^p}{q^p}\left(1 + \frac{p}{q}\right)^{-q} \\ &\sim \frac{p!}{q^p}, \qquad q \to \infty \end{aligned} \tag{5.4.15}$$

Recall that R does not appear in the quasi-elastic solution. Except for purely elastic materials ($p = 0$), the quasi-elastic solution is a very bad approximation for acceleration waves at small times when the material is strongly nonlinear at small strains. However, for small q (large γ), expansion of the factorials yields

$$R \sim 1 - \left(\Upsilon + \frac{(p!)'}{p!}\right) q, \qquad q \to 0, \tag{5.4.16}$$

where $(p!)'$ is the derivative of the factorial function evaluated at p and $\Upsilon = 0.5772\ldots$ is Euler's constant [5.1]. The quasi-elastic solution more closely approximates the solution at small times when the material is weakly nonlinear at small strains.

5.5 Numerical Examples.

In section 5.4 we presented asymptotic results for acceleration waves which are exact for a material having a specific strain curve and a power-law compliance. Such a compliance grows without bound and cannot represent a viscoelastic solid for all time. To better illustrate the global behaviour of acceleration waves in solids, we consider the steady acceleration wave problem for a material having the normalized *exponential compliance*

$$\chi(t) = \chi_e \left(1 - e^{(-t/t_r)}\right) H(t), \tag{5.5.1}$$

with the strain curve

$$f(\varepsilon) = \varepsilon + k\varepsilon^{\gamma}, \quad k > 0, \quad \gamma > 1. \tag{5.5.2}$$

In terms of the compliance J, the equilibrium value χ_e appearing in (5.5.1) is given by:

$$\chi_e = (J_e - J_o)/J_o. \tag{5.5.3}$$

We use a simple numerical approximation to the iteration scheme (4.6.8) to generate approximations to a finite number, K, of terms belonging to upper and lower bound sequences for the scaled strain of equation (4.6.5). We similarly generate such sequences for shocks in materials having either a power-law or exponential compliance combined with this strain curve.

The governing equation for the acceleration wave problem is

$$(u(t))^{\gamma} = (\chi' \star u)(t). \tag{5.5.4}$$

The iteration scheme takes the form

$$u_{k+1} = \Big((\chi' \star u_k)(t)\Big)^{1/\gamma}; \qquad k = 0, 1, 2, \ldots, K-1. \tag{5.5.5}$$

The bounding sequences are desired for $t \in [0, T]$.

In order to perform the convolution on the iterates $\{u_k\}_{k=1}^{K-1}$, we use a piecewise linear approximation to u_k and convolve this approximation exactly with χ. We partition the interval $[0, T]$ into N subintervals of equal length given by

$$h = T/N. \tag{5.5.6}$$

The piecewise linear approximation to a function φ on $[0, T]$ is

$$L_T^N\, \varphi(t) = \varphi_i + \frac{1}{h}(\varphi_{i+1} - \varphi_i)\cdot(t - t_i), \qquad t_i \le t \le t_{i+1}. \tag{5.5.7}$$

where

$$t_i = i\,h \quad \text{and} \quad \varphi_i = \varphi(t_i) \quad \text{for} \quad i = 0, 1, 2, \ldots, N. \tag{5.5.8}$$

The iteration scheme for the numerical approximations is

$$u_{k+1}^N = L_T^N\, (\chi' \star u_k^N)^{1/\gamma}(t), \qquad k = 0, 1, 2, \ldots, K-1, \tag{5.5.9}$$

where $u_k^N(t) = L_T^N\, u_k(t)$. The upper bound and lower bound sequences we generate are started with

$$u_o^N(t) = L_T^N\, u_Q(t) \tag{5.5.10}$$

and

$$u_o^N(t) = L_T^N\, u_L(t), \tag{5.5.11}$$

respectively, where u_Q is the quasi-elastic solution given by

$$u_Q(t) = (\chi(t))^{\frac{1}{\gamma-1}}, \qquad t \in [0, T], \tag{5.5.12}$$

and u_L is the lower bound from equation (4.6.13):

$$u_L(t) = r\, t^{\frac{1}{\gamma-1}}, \qquad t \in [0, T]. \tag{5.5.13}$$

For the strain curve considered, the constant r is r_1 in the proof of Lemma 4.6.2:

$$r = \left(\frac{\chi(T)}{T}\frac{\gamma - 1}{\gamma}\right)^{\frac{1}{\gamma-1}}, \tag{5.5.14}$$

with the aid of equation (4.6.11).

From the numerical scheme, we generate the iterates u_k^N at the mesh points t_i. Let φ^N be any one of the piecewise linear u_k^N. Its convolution at t_i is:

$$\begin{aligned}(\chi' \star \varphi^N)(t_i) &= \int_0^{t_i} \varphi^N(t_i - \tau)\, d\chi(\tau) \\ &= \sum_{k=1}^{i} I_k, \end{aligned} \tag{5.5.15}$$

where the k-th integral in the summation is given by

$$\begin{aligned} I_k &= \int_{t_{k-1}}^{t_k} \varphi^N(t_i - \tau)\, d\chi(\tau) \\ &= \int_{t_{k-1}}^{t_k} \left\{ \varphi_{i-k}^N + \frac{\varphi_{i-k+1}^N - \varphi_{i-k}^N}{h}(t_k - \tau)\right\} d\chi(\tau), \end{aligned} \tag{5.5.16}$$

in view of (5.5.7) and with the notation of (5.5.8). We put our results in terms of the non-dimensional time t/t_r appearing in equation (5.5.1) for the exponential compliance. With the change of variables $t/t_r \to t$, we have $d\chi(\tau) = \chi_e e^{-\tau}\, d\tau$. Using this to integrate (5.5.16) exactly, we obtain after simplification:

$$\begin{aligned} I_k = &\left[(1-k)\varphi_{i-k}^N + k\,\varphi_{i-k+1}^N\right]\cdot(\chi_k - \chi_{k-1}) \\ &+ \frac{\chi_e}{h}(\varphi_{i-k+1}^N - \varphi_{i-k}^N)\cdot\left[(1+kh)e^{-kh} - (1+(k-1)h)e^{-(k-1)h}\right]. \end{aligned} \tag{5.5.17}$$

The numerical scheme is now completely defined. It consists of equations (5.5.9), (5.5.15), (5.5.17), and either (5.5.10) or (5.5.11) to get started.

We remark that we chose this approximation to the exact iteration scheme both for simplicity and since any continuous function can be approximated arbitrarily closely by piecewise linear functions [5.2]. For a finite number of terms generated by this numerical scheme, we can get arbitrarily close to the exact iterates by taking N large enough.

We present the results of this scheme as a series of graphs of approximations to acceleration waves for three values of the nonlinearity γ. In Figures 5.1–5.4 we have plotted the normalized iterates $u_k \chi_e^{(1/1-\gamma)}$ vs. t/t_r. Figure 5.1 illustrates truncated upper bound and lower bound sequences for $\gamma = 5/4$. It shows iterates 1–25 of the upper bound (decreasing) sequence started by u_Q and iterates 1–30 of the lower bound (increasing) sequence generated by u_L. For each curve in this figure, 201 points were used in the computations, whereas, 101 are plotted. The large space between the truncated sequences is the numerical approximation to a bound within which the acceleration wave is to be found. Our approximation to this acceleration wave is illustrated in Figure 5.2 along with the quasi-elastic solution for $\gamma = 5/4$. The approximate acceleration wave is the superposition of the indistinguishable 47-th members of the upper and lower bound sequences of the previous figure. Figures 5.3 and 5.4 illustrate the quasi-elastic solutions and similar approximations to the acceleration waves for $\gamma = 2$ and $\gamma = 5$, respectively. In Figures 5.2–5.4, all 201 points used in the computation of each curve are plotted. We note that the exponential compliance is regularly-varying as $t \downarrow 0$ with power $p = 1$. The graphs for these three values of γ illustrate approximations for acceleration waves which are regularly-varying for $t \downarrow 0$ with power $q = 1/(\gamma - 1)$; these waves have initial slopes which are zero ($\gamma = 5/4$), finite ($\gamma = 2$), and infinite ($\gamma = 5$). We recall from the previous section that the quasi-elastic solution is asymptotically incorrect by a multiplicative factor, for $t \downarrow 0$.

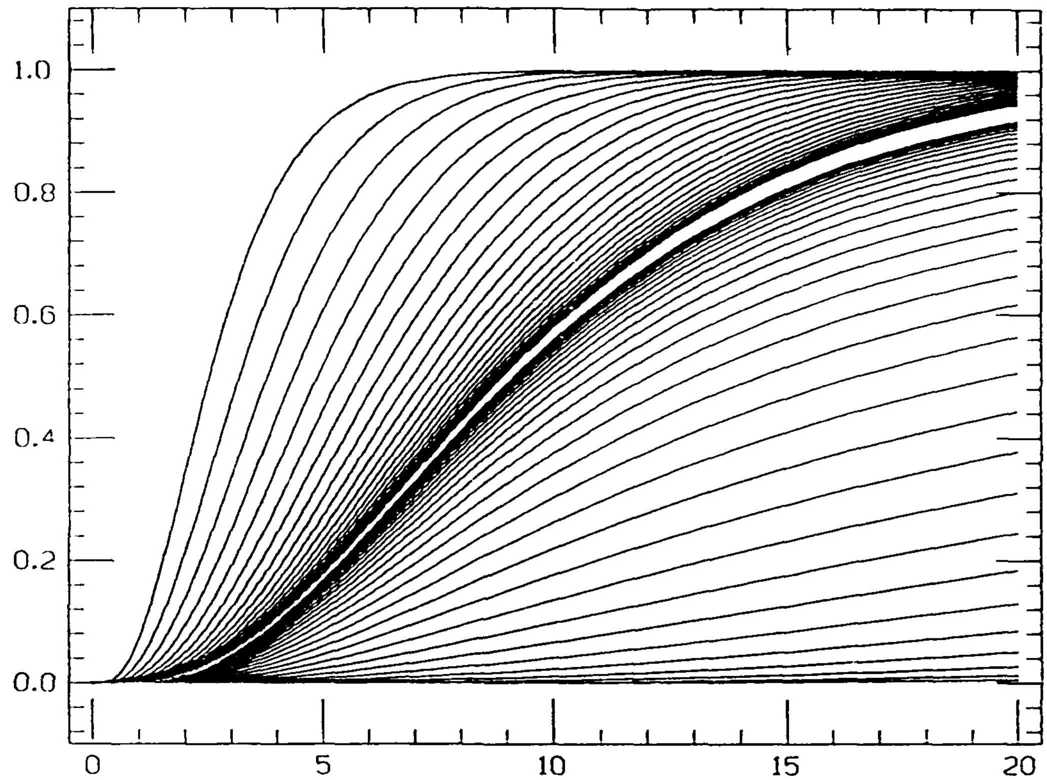

Figure 5.1. Approximations to bounding sequences for the strain acceleration wave for $\gamma = 5/4$ and an exponential compliance. The ordinate variable is $(k/\chi_e)^{(1/\gamma-1)}\varepsilon$. The abscissa represents t/t_r.

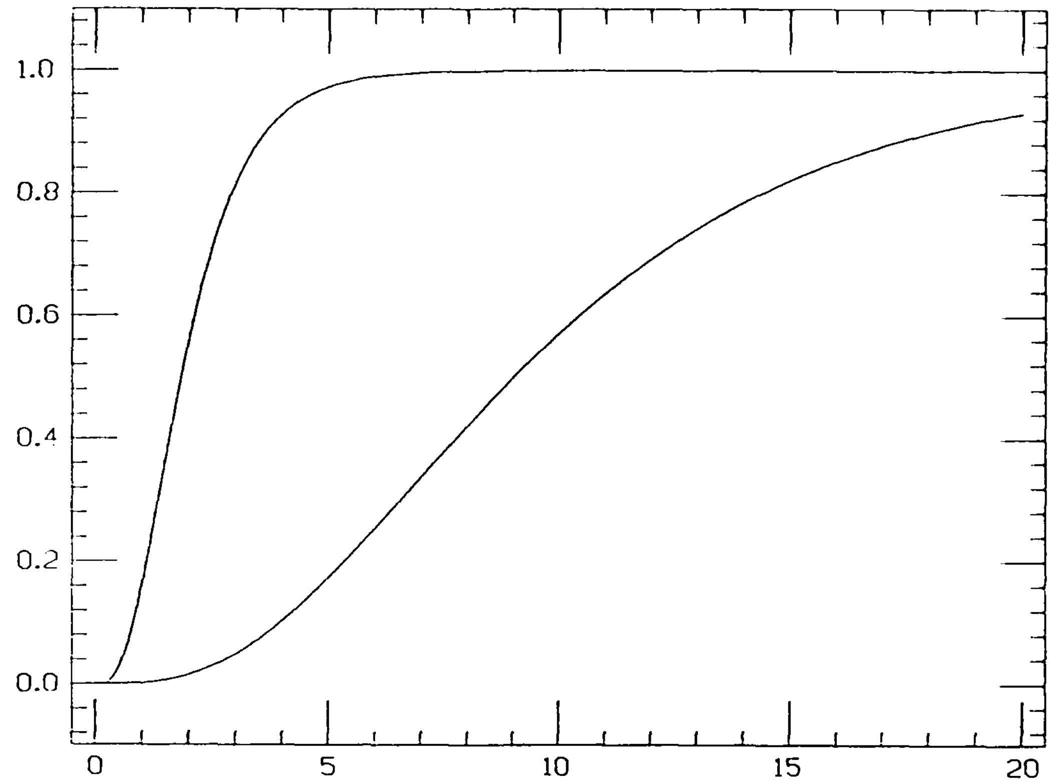

Figure 5.2. The quasi-elastic solution (top curve) and the superimposed 47-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain acceleration wave with $\gamma = 5/4$ and an exponential compliance. The ordinate variable is $(k/\chi_e)^{(1/\gamma-1)}\varepsilon$. The abscissa represents t/t_r.

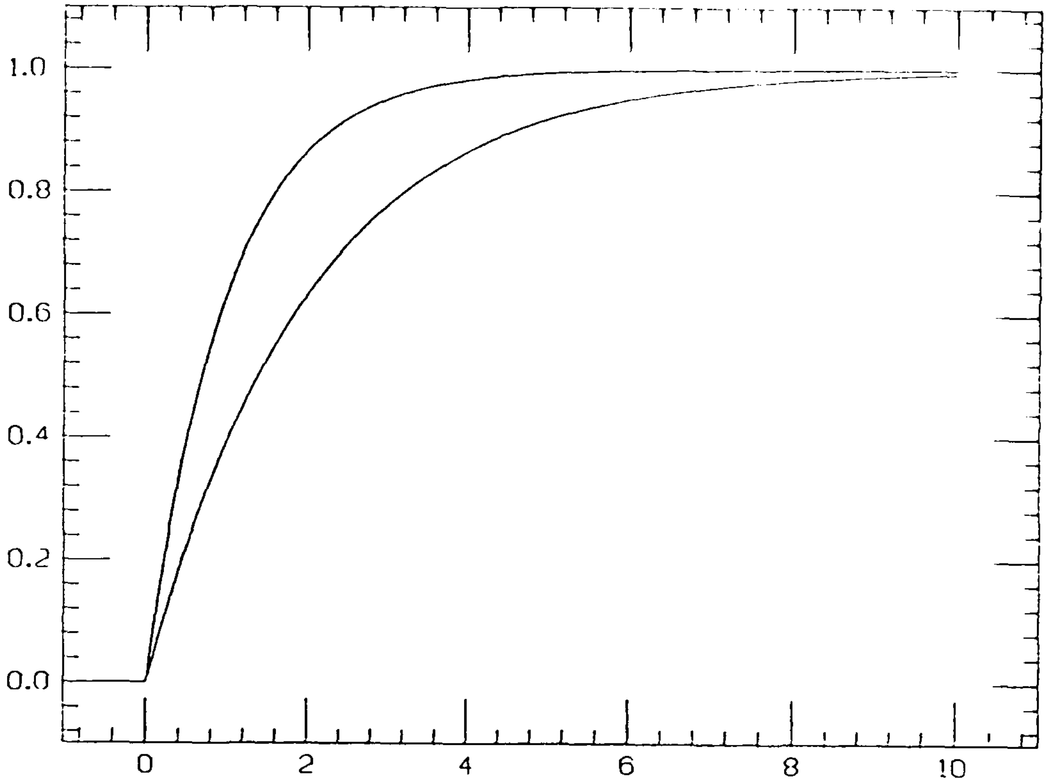

Figure 5.3. The quasi-elastic solution (top curve) and the superimposed 15-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain acceleration wave with $\gamma = 2$ and an exponential compliance. The ordinate variable is $(k/\chi_e)^{(1/\gamma-1)}\varepsilon$. The abscissa represents t/t_r.

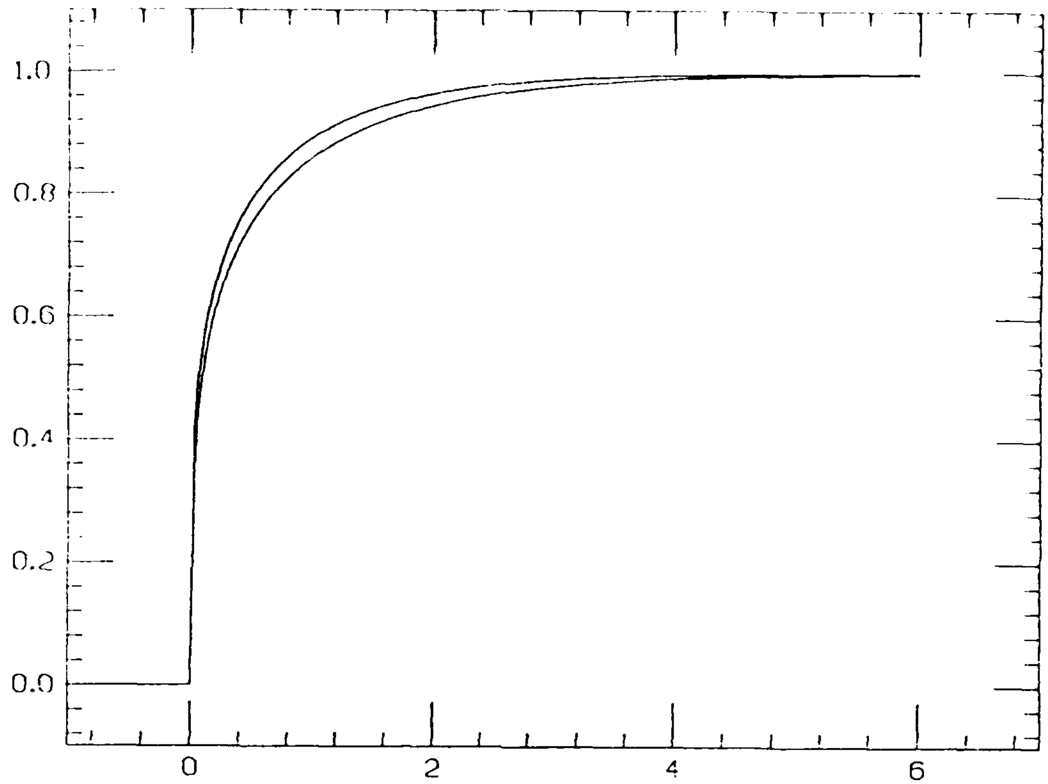

Figure 5.4. The quasi-elastic solution (top curve) and the superimposed 6-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain acceleration wave with $\gamma = 5$ and an exponential compliance. The ordinate variable is $(k/\chi_e)^{(1/\gamma-1)}\varepsilon$. The abscissa represents t/t_r.

We now consider problems for steady shocks in materials with strain curves of the form (5.5.2). Our results are given in terms of $\overline{u}$, the scaled strain of equation (4.6.5) normalized by its value at the discontinuity:

$$\overline{u}(t) = u(t)/u_o = \varepsilon(t)/\varepsilon_o. \tag{5.5.18}$$

After we derive an equaton for $\overline{u}$, we present graphs of numerical approximations to its solutions based upon the scheme presented above for convolutions with an exponential normalized compliance. We then modify the scheme for power-law normalized compliances and present the graphs of approximations for shocks in such materials.

To obtain the governing equation for $\overline{u}$, we cancel $f(\varepsilon_o) = \lambda\varepsilon_o$ from both sides of equation (5.3.2) and use (5.5.2) explicitly for f:

$$k(\varepsilon(t))^{\gamma} = (\lambda - 1)\varepsilon(t) + \lambda(\chi' \star \varepsilon)(t). \tag{5.5.19}$$

(Recall $\lambda = (U/U_o)^2$.) In terms of the scaled strain, this is:

$$(u(t))^{\gamma} = (\lambda - 1)u(t) + \lambda(\chi' \star u)(t). \tag{5.5.20}$$

We note that this equation reduces to (5.5.4) governing acceleration waves at $\lambda = 1$. For $\lambda > 1$, the jump in the shock wave has the value:

$$u_o = u(0) = (\lambda - 1)^{\frac{1}{\gamma - 1}}, \tag{5.5.21}$$

since $\chi(0) = 0$ implies that the convolution in (5.5.20) vanishes at $t = 0$. When we divide (5.5.20) by u_o^{γ}, we obtain:

$$(\overline{u}(t))^{\gamma} = \overline{u}(t) + (\overline{\chi}' \star \overline{u})(t), \tag{5.5.22}$$

where

$$\overline{\chi}(t) = \frac{\lambda}{\lambda - 1}\chi(t). \tag{5.5.23}$$

The corresponding iteration scheme for K terms in a sequence is

$$\bar{u}_{k+1}(t) = \left(\bar{u}_k(t) + (\bar{\chi}' \star \bar{u}_k)(t)\right)^{1/\gamma}, \qquad k = 0, 1, 2, \ldots, K-1 \tag{5.5.24}$$

We again choose the quasi-elastic solution to start our upper bound sequences.

$$\bar{u}_o(t) = \bar{u}_Q(t) = \left(H(t) + \bar{\chi}(t)\right)^{\frac{1}{\gamma-1}}. \tag{5.5.25}$$

It is easy to show that a starter for a lower bound sequence is the Heaviside step function:

$$\bar{u}_o(t) = \bar{u}_L(t) = H(t). \tag{5.5.26}$$

For the exponential compliance in equation (5.5.1), our normalization of the solution by its jump value requires specification of the parameter

$$c = \frac{\lambda}{\lambda - 1}\chi_e. \tag{5.5.27}$$

For any choice of $c > \chi_e$ the normalized wavespeed being considered is:

$$\lambda = \frac{c}{c - \chi_e}. \tag{5.5.28}$$

In what follows, we have chosen $c = 1$ to illustrate shock behaviour depending on the nonlinearity γ. This necessarily restricts the compliance to have an equilibrium value $\chi_e < 1$; however, this is not a severe restriction. Materials for which $\chi_e \ll 1$ are nearly elastic; with small values of χ_e, the graphs in Figures 5.5–5.8 represent numerical approximations to shocks at wavespeeds just above the acceleration wavespeed in nearly elastic materials. Furthermore, values of χ_e approaching unity represent more strongly viscoelastic solids having equilibrium compliances near $J_e = 2J_o$, according to (5.5.3). For such materials, the graphs represent high speed cases, $\lambda \gg 1$.

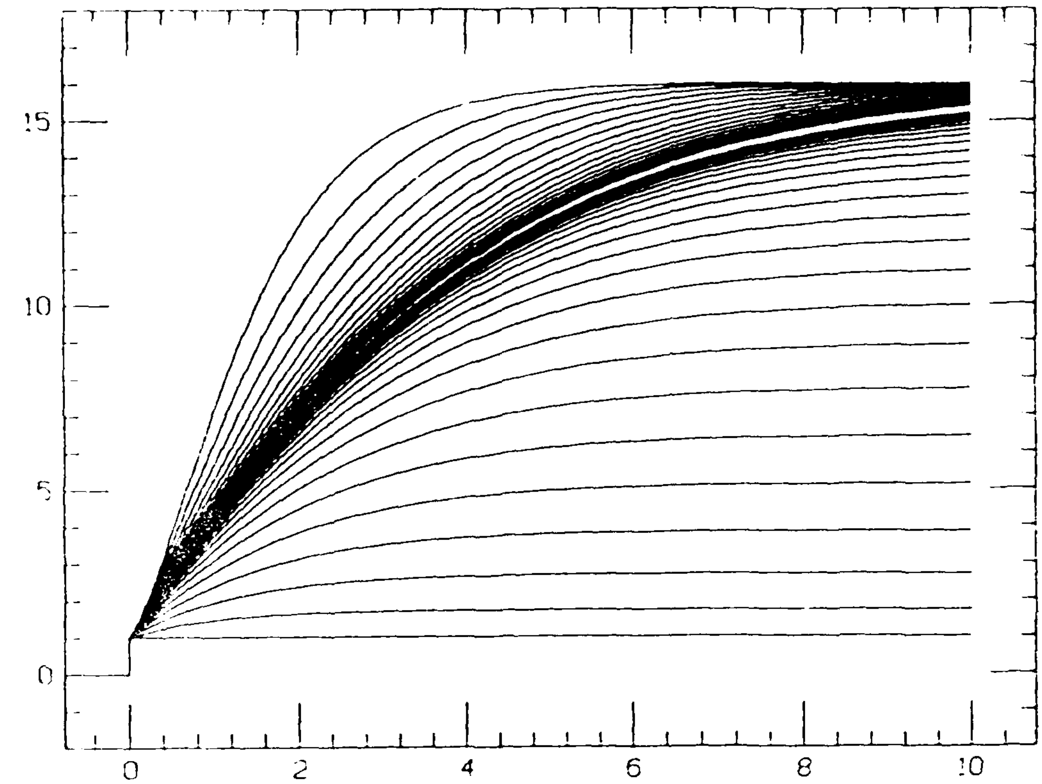

Figure 5.5. Approximations to bounding sequences for the strain shock wave for $\gamma = 5/4$, an exponential compliance, and $c = 1$. The ordinate variable is $\varepsilon/\varepsilon_0$. The abscissa represents t/t_r.

Figure 5.5 illustrates truncated upper bound and lower bound shock sequences for $\gamma = 5/4$, analogous to Figure 5.1 for acceleration waves. Here we show $\bar{u}_0$ and the first 17 members of the decreasing sequence it starts, along with the step function, u_L, and the first 25 members of its increasing sequence. Again, 201 points were used in computation and 101 are plotted for each curve. Figures 5.6–5.8 illustrate the quasi-elastic shock solutions and approximations to the shock solutions for $\gamma = 5/4$, 2, and 5, respectively. For these graphs, all 201 computed points are plotted in each curve. For shocks, the quasi-elastic solution is asymptotic to the solution for $t \downarrow 0$ (§§5.1). It is evident from these figures that the quasi-elastic solution is acceptable as an approximation for a quadratic nonlinearity with this compliance; it is very good for $\gamma = 5$. However, for $\gamma = 5/4$, it is no better than a loose upper bound, except in the small-time asymptotic region.

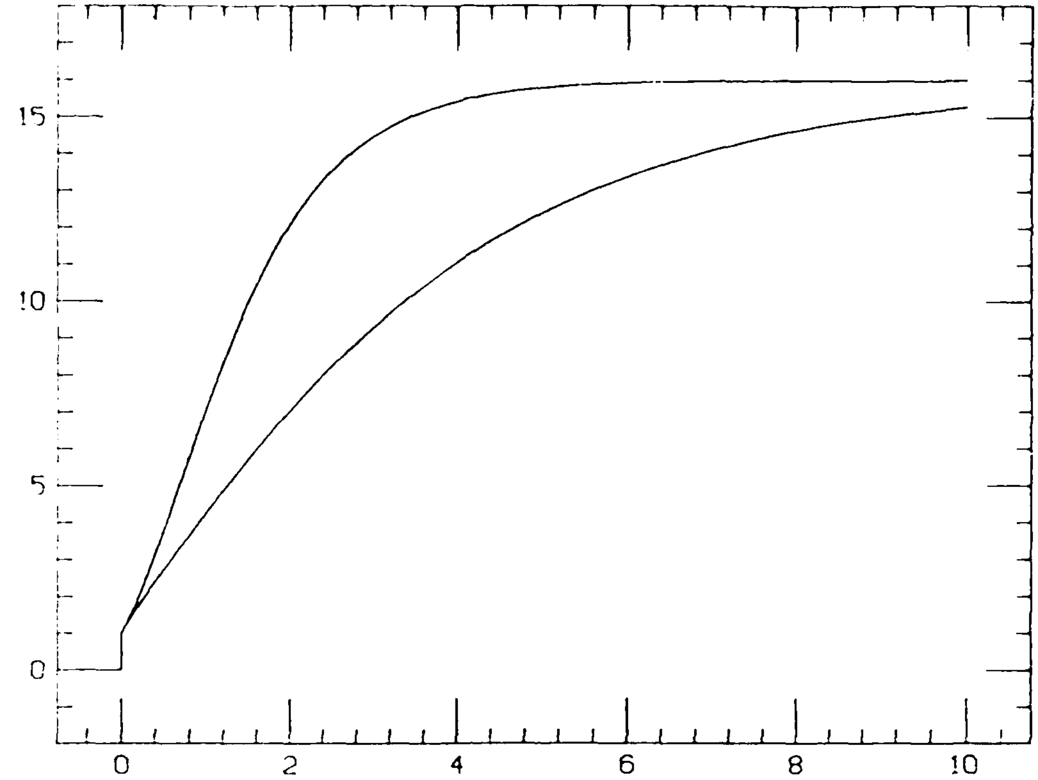

Figure 5.6. The quasi-elastic solution (top curve) and the superimposed 45-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5/4$, an exponential compliance. and $c = 1$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents t/t_r.

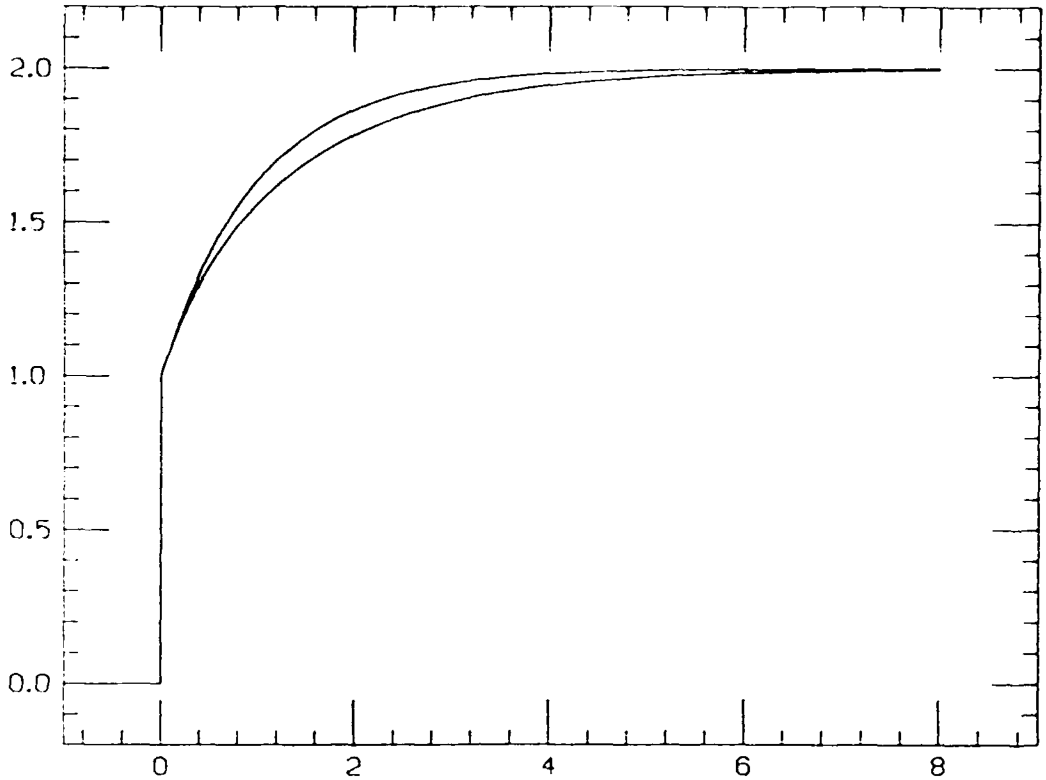

Figure 5.7. The quasi-elastic solution (top curve) and the superimposed 14-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 2$, an exponential compliance, and $c = 1$. The ordinate variable is $\varepsilon/\varepsilon_0$. The abscissa represents t/t_r.

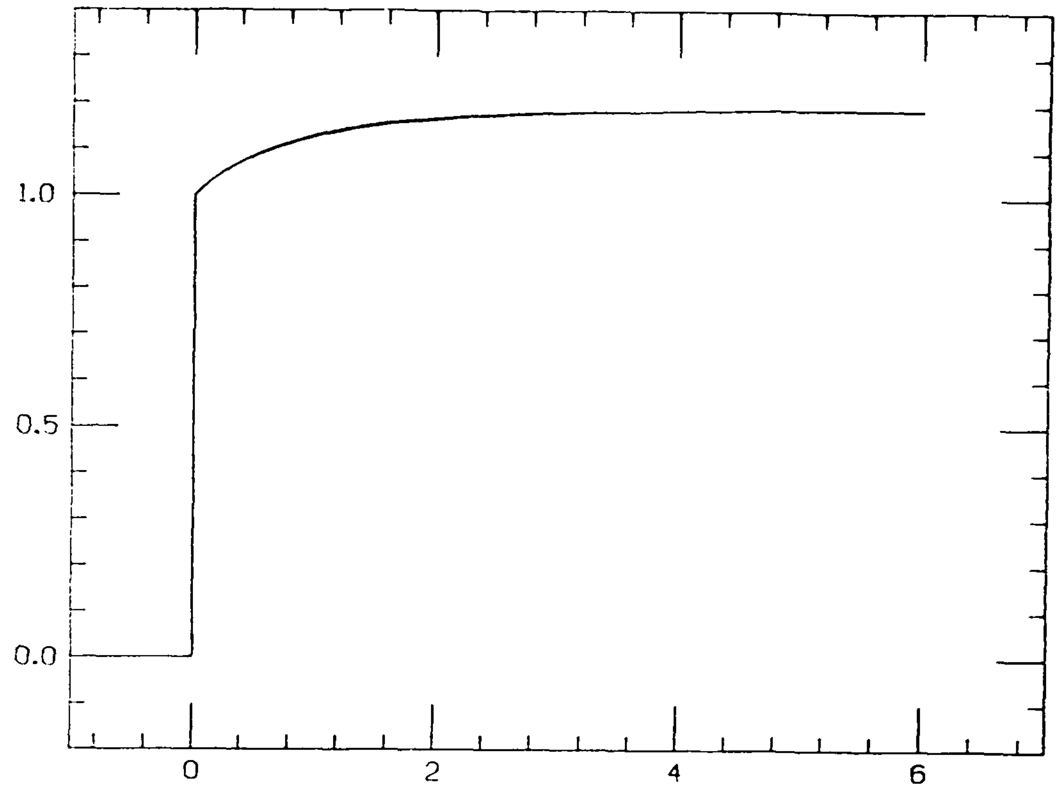

Figure 5.8. The quasi-elastic solution and the superimposed 6-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5$, an exponential compliance, and $c = 1$. The ordinate variable is $\varepsilon/\varepsilon_0$. The abscissa represents t/t_r.

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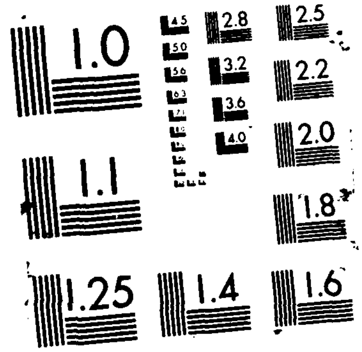
1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
6.3

For the exponential compliance considered thus far, we could have successfully used a piecewise linear approximation to the entire integrand in the convolution; i.e., the familiar trapezoid quadrature rule [5.3]. We chose to integrate the iterates exactly with $d\chi$ in anticipation of the singular problems which we now consider, involving the power-law normalized compliance:

$$\chi(t) = \chi_o t^p, \qquad p \in [0,1]. \tag{5.5.29}$$

The numerical scheme is modified by the evaluation of the integrals I_k in equation (5.5.16). The non-dimensional time in this case is $\chi_o^{1/p} t$. We make the change of variables $\chi_o^{1/p} t \rightarrow t$ in (5.5.16) and use $d\chi(\tau) = p\tau^{p-1}\,d\tau$. The result for I_k after simplification is:

$$\begin{aligned} I_k = &\left[(1-k)\varphi_{i-k}^N + k\,\varphi_{i-k+1}^N\right]\cdot(\chi_k - \chi_{k-1}) \\ &- (\varphi_{i-k+1}^N - \varphi_{i-k}^N)\frac{p}{p+1}h^p\left(k^{p+1} - (k-1)^{p+1}\right). \end{aligned} \tag{5.5.30}$$

For the numerical scheme using (5.5.30), we again generate upper bound and lower bound sequences which are started, respectively, by $L_T^N\,\overline{u}_Q$ and $L_T^N\,\overline{u}_L$. The resulting approximate solutions for shocks in a nonlinear power-law material are illustrated in Figures 5.9–5.17. These figures include all combinations of nonlinearity $\gamma = 5/4$, 2, and 5 with compliance powers $p = 1/10$, $1/2$, and $9/10$. For these figures, we used 401 points in the computations, all of which are plotted for each curve. We have removed the wavespeed from the the problem by plotting $\overline{u}$ against the non-dimensional time $[\lambda\chi_o/(\lambda-1)]^{1/p}t$. It is apparent from these figures that, for the problems considered, the quasi-elastic solution is better as an approximation to the solution in a power-law material when the nonlinearity γ is large, or when the power p is small, or both.

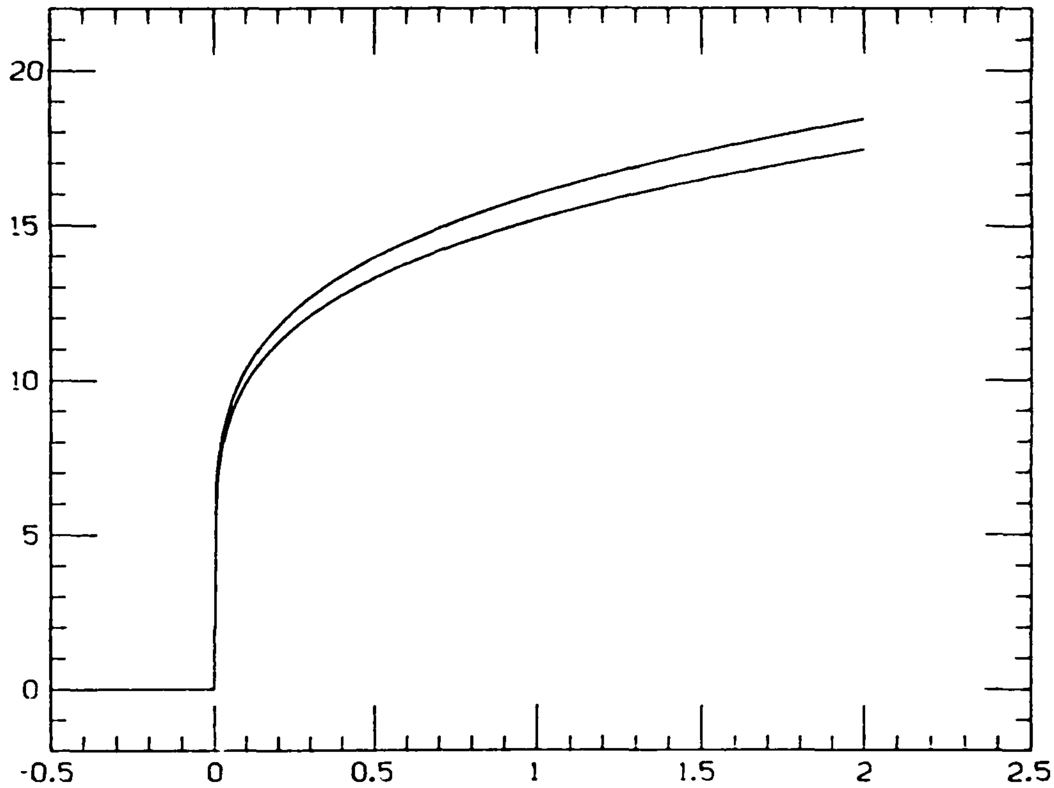

Figure 5.9. The quasi-elastic solution (top curve) and the superimposed 50-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5/4$ and $p = 1/10$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

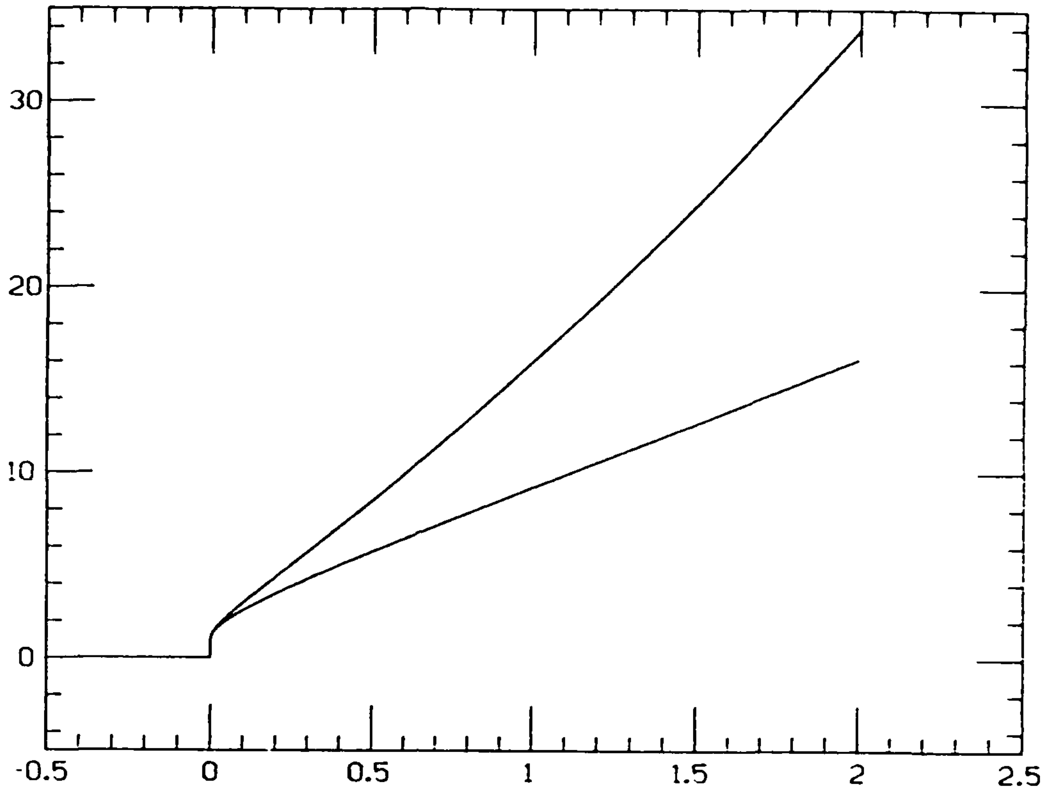

Figure 5.10. The quasi-elastic solution (top curve) and the superimposed 46-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5/4$ and $p = 1/2$. The ordinate variable is ϵ/ϵ_o. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

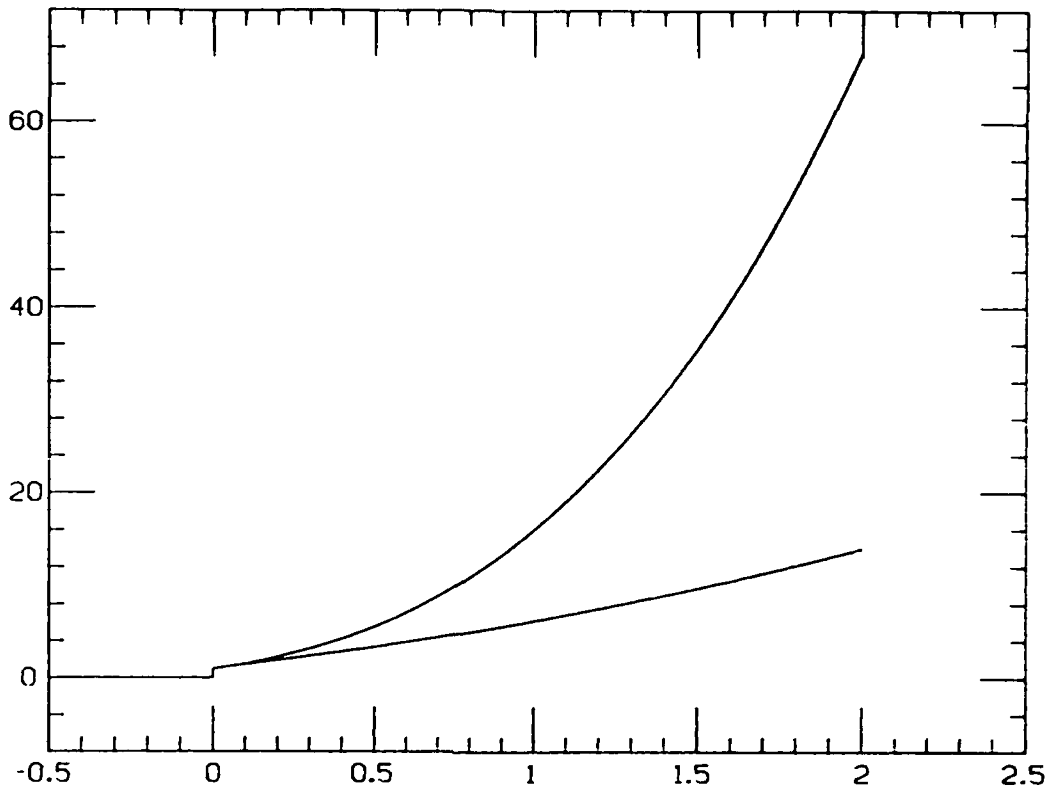

Figure 5.11. The quasi-elastic solution (top curve) and the superimposed 41-st iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5/4$ and $p = 9/10$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

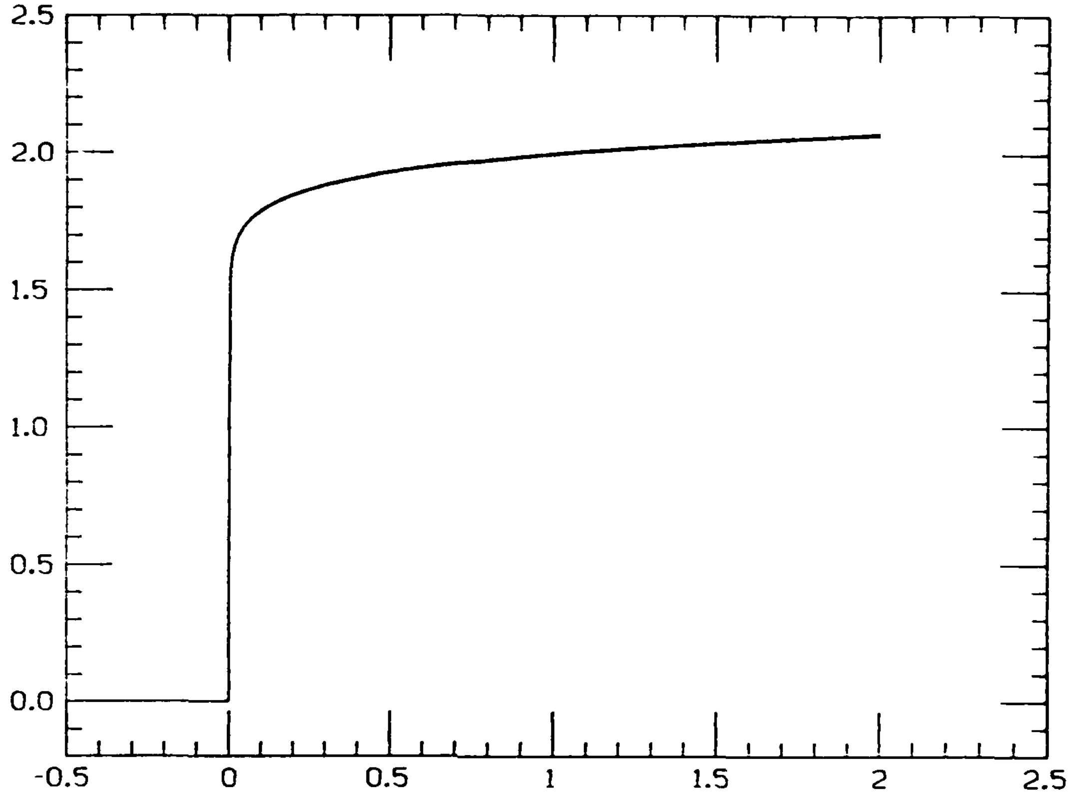

Figure 5.12. The quasi-elastic solution and the superimposed 14-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 2$ and $p = 1/10$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

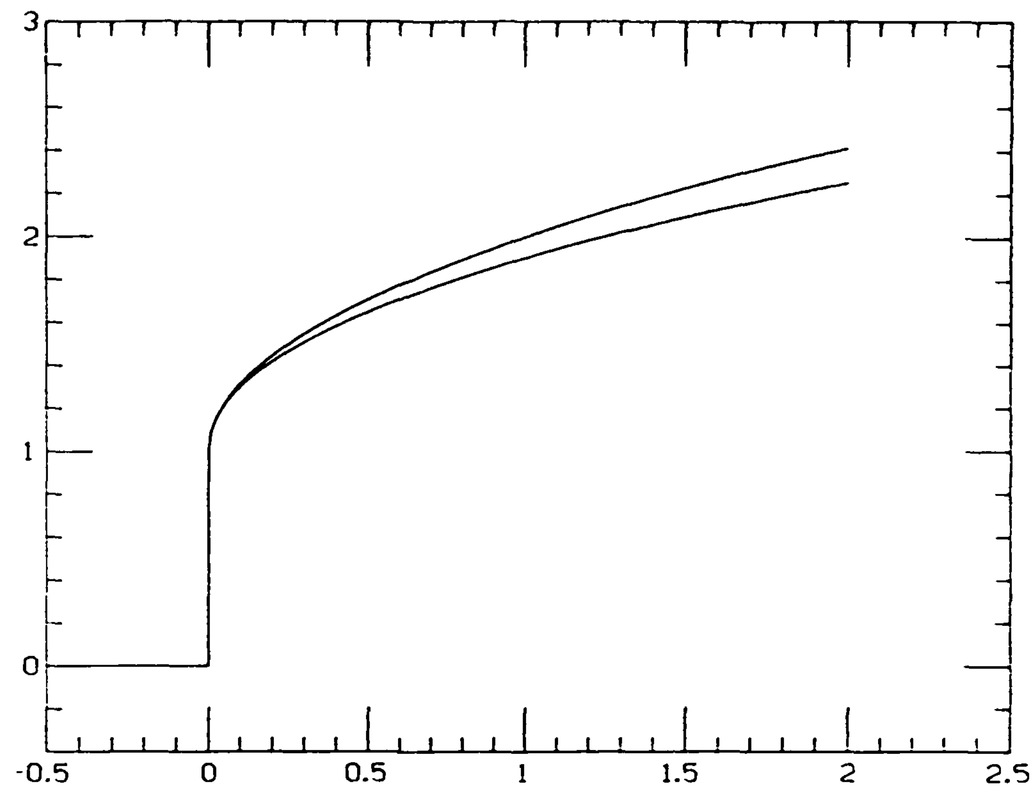

Figure 5.13. The quasi-elastic solution (top curve) and the superimposed 14-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 2$ and $p = 1/2$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

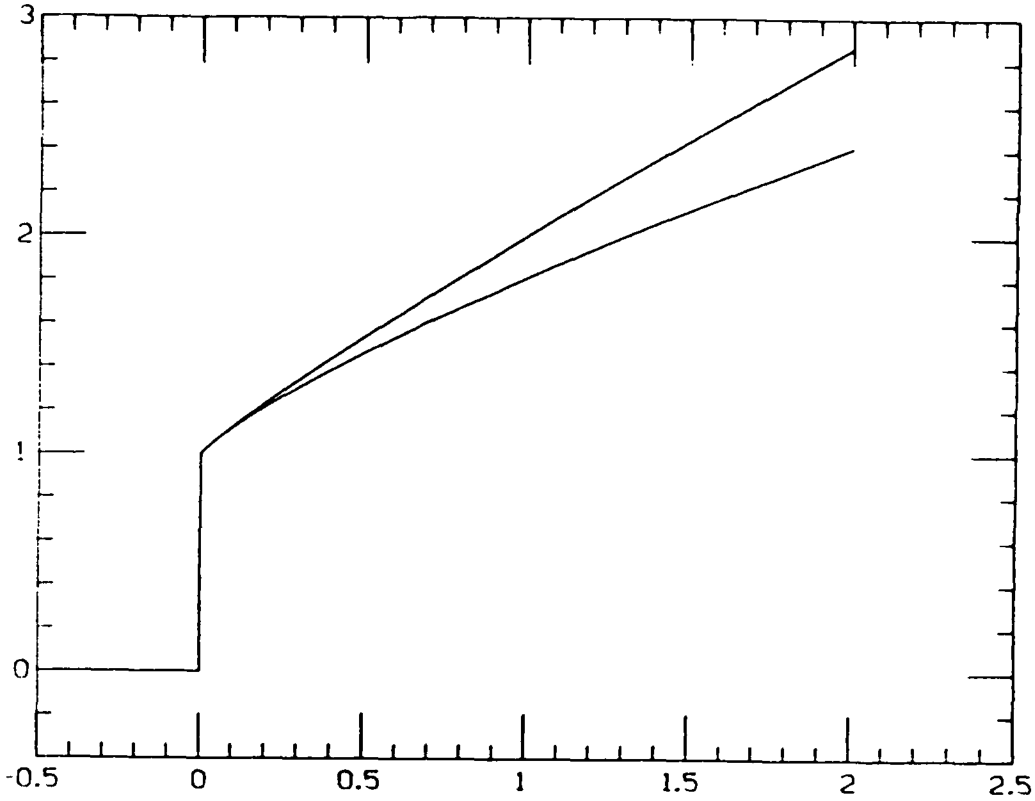

Figure 5.14. The quasi-elastic solution (top curve) and the superimposed 13-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 2$ and $p = 9/10$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_c/(\lambda - 1)]^{1/p_t}$.

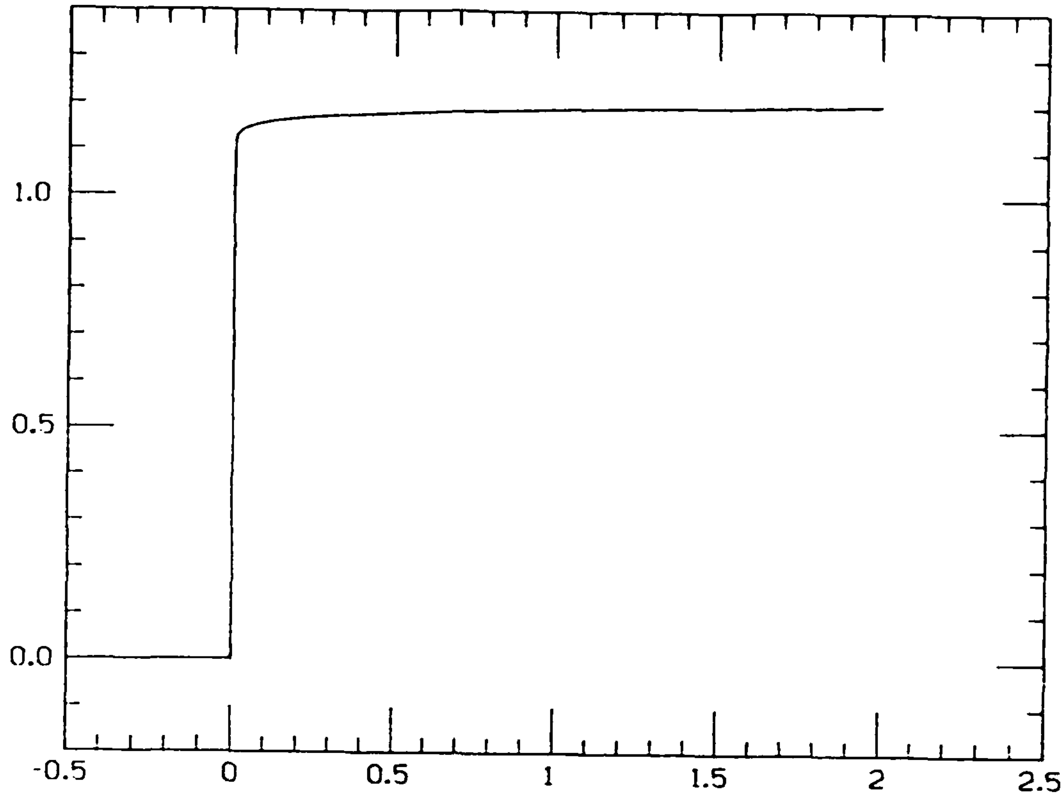

Figure 5.15. The quasi-elastic solution and the superimposed 6-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5$ and $p = 1/10$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

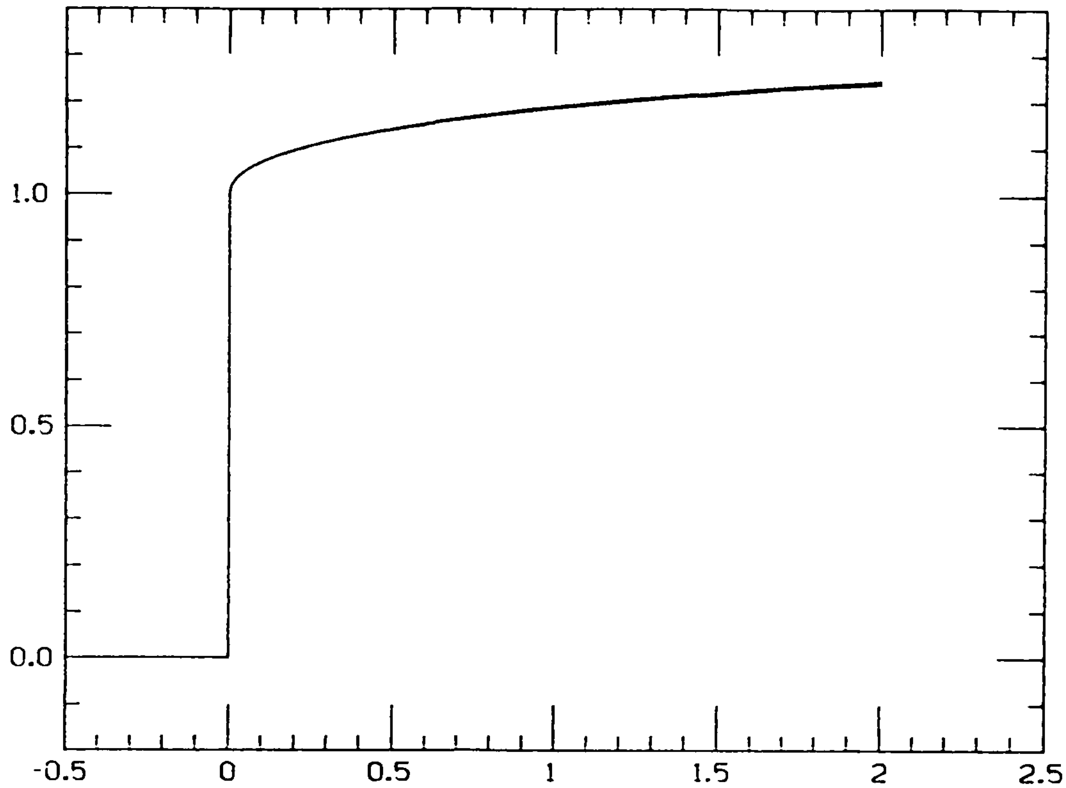

Figure 5.16. The quasi-elastic solution and the superimposed 6-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5$ and $p = 1/2$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

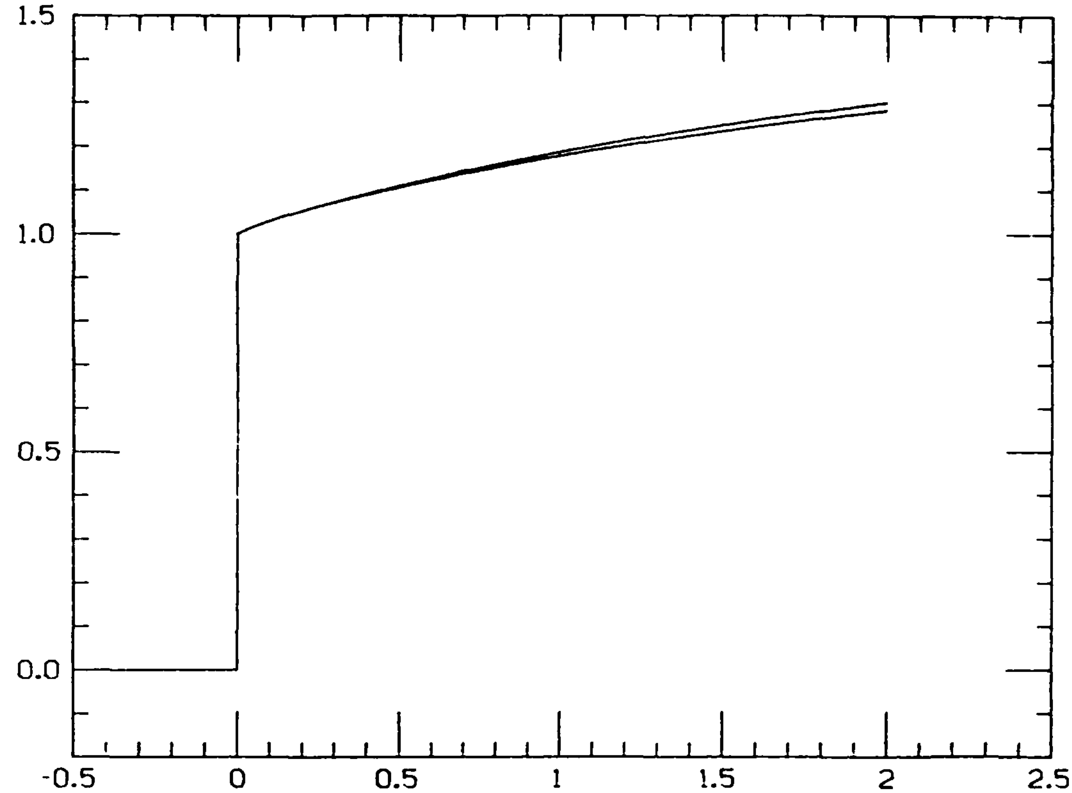

Figure 5.17. The quasi-elastic solution (top curve) and the superimposed 6-th iterates in the upper and lower bound sequences representing the approximate numerical solution for the strain shock wave with $\gamma = 5$ and $p = 9/10$. The ordinate variable is $\varepsilon/\varepsilon_o$. The abscissa represents $[\lambda\chi_o/(\lambda - 1)]^{1/p}t$.

CHAPTER 6: SOLUTIONS BELOW THE ACCELERATION WAVESPEED

6.1 Prelude.

In this chapter, we consider problems for non-negative steady strain waves, $\varepsilon(t)$, traveling at speeds slower than that of the acceleration wave, i.e., for $U < U_o$. We have shown in Chapter 3 (§§3.3) that there are no non-negative non-trivial steady waves which travel at or below the equilibrium wavespeed U_e. From Chapter 4 (§§4.5), we know that when $U \in (U_e, U_o)$ there are continuous solutions $\varepsilon(t)$ which are asymptotic to e^{rt} as $t \to -\infty$ with a value of r which depends on U and the material compliance. In section 6.2, we consider the problem for such a solution when the wavespeed is perturbed from the limiting value U_e. We denote by δ a small parameter and obtain solutions $\varepsilon(t;\delta)$ which are asymptotically correct as $\delta \downarrow 0$ when $U^2 = U_e^2(1+\delta)$. In section 6.3 we produce numerical solutions for all wavespeeds $U < U_o$ in power law materials with quadratic nonlinearity.

6.2 $U^2 = U_e^2(1+\delta)$.

We first derive the leading term in an asymptotic series for $\varepsilon(t;\delta)$ as $\delta \downarrow 0$ when the strain curve $f(\varepsilon)$ is specified no more completely than in equation (2.3.18). In order to obtain the next term in such a series, or to just estimate the size of the next term, we need to further specify f. We do so in the rest of this section which we devote to strain curves which are accurately represented by a power series for small ε. For these materials, we obtain the first two terms in the asymptotic series for ε along with an estimate of the size of the third term.

We recall that the governing equation is:

$$f(\varepsilon) = U^2(J' \star \varepsilon), \tag{6.2.1}$$

subject to

$$\varepsilon(-\infty) = 0. \tag{6.2.2}$$

We leave implicit the dependence of ε on δ and we seek a solution ε such that

$$\varepsilon(t) \sim e^{rt}, \qquad t \to -\infty, \tag{6.2.3}$$

where r satisfies

$$U^2 r \overline{J}(r) = 1, \tag{6.2.4}$$

according to equations (4.5.1), (3.2.3), and (3.2.4). We recall from equations (3.2.5) and the discussion following them, that $r \downarrow 0$ as $U \downarrow U_e$. Thus, r is a small parameter when δ is.

We proceed with expansions of both sides of (6.2.1) which will eventually result in an equation containing powers of the small parameter δ from which we can obtain equations for the terms in an asymptotic series for ε. Equation (6.2.3) suggests the scaled time variable:

$$\theta = rt. \tag{6.2.5}$$

Throughout, we denote by hatted variables functions of θ. For example,

$$\begin{aligned} \varepsilon(t) &= \hat{\varepsilon}(\theta) \\ J(t) &= \hat{J}(\theta), \end{aligned} \tag{6.2.6}$$

and so on. Using equations (2.5.3) and (6.2.5), the convolution is:

$$\begin{aligned} (J' \star \varepsilon)(t) &= (J \star \varepsilon')(t) \\ &= (\hat{J} \star \hat{\varepsilon}')(\theta) \\ &= \int_{-\infty}^{\theta} \hat{J}(\theta - \eta)\, d\hat{\varepsilon}(\eta) \\ &= \int_{-\infty}^{\theta} \hat{J}(\theta - \eta)\hat{\varepsilon}'(\eta)\, d\eta, \end{aligned} \tag{6.2.7}$$

where we have used the continuity of ε at these wavespeeds (§§3.3) in the last line of (6.2.7). Thus, the governing equation (6.2.1) becomes:

$$f(\hat{\varepsilon}(\theta)) = U^2(\hat{J} \star \hat{\varepsilon}')(\theta). \tag{6.2.8}$$

The change of variables $\theta - \eta \to \eta$ in the integral yields

$$(\hat{J} \star \hat{\varepsilon}')(\theta) = \int_0^\infty \hat{\varepsilon}'(\theta - \eta)\hat{J}(\eta)\, d\eta. \tag{6.2.9}$$

We use (5.1.8) to write

$$\hat{J}(\theta) = J_e[H(\theta) - \hat{\psi}(\theta)] \tag{6.2.10}$$

in equation (6.2.9) and evaluate the first integral to obtain:

$$(\hat{J} \star \hat{\varepsilon}')(\theta) = J_e\hat{\varepsilon}(\theta) - J_e\int_0^\infty \hat{\varepsilon}'(\theta - \eta)\hat{\psi}(\eta)\, d\eta. \tag{6.2.11}$$

We assume that ε is sufficiently differentiable to do the following. We expand $\hat{\varepsilon}'(\theta - \eta)$ about θ and obtain:

$$(\hat{J} \star \hat{\varepsilon}')(\theta) = J_e\hat{\varepsilon}(\theta) - J_e\int_0^\infty \left[\hat{\varepsilon}'(\theta) - \eta\hat{\varepsilon}''(\theta) + \frac{1}{2!}\eta^2\hat{\varepsilon}'''(\theta) - + \cdots\right]\hat{\psi}(\eta)\, d\eta. \tag{6.2.12}$$

Let us define

$$\psi_j = \int_0^\infty \tau^j\psi(\tau)\, d\tau; \qquad j = 0, 1, 2, \dots, \tag{6.2.13}$$

so far as these moments exist. Then,

$$\int_0^\infty \eta^j\hat{\psi}(\eta)\, d\eta = r^{j+1}\psi_j. \tag{6.2.14}$$

Using (6.2.13) and (6.2.14) in (6.2.12), we have:

$$(\hat{J} \star \hat{\varepsilon}')(\theta) = J_e\left[\hat{\varepsilon}(\theta) - r\psi_0\hat{\varepsilon}'(\theta) + r^2\psi_1\hat{\varepsilon}''(\theta) - \frac{1}{2!}r^3\psi_2\hat{\varepsilon}'''(\theta) + - \cdots\right]. \tag{6.2.15}$$

If the moments ψ_j fail to exist beyond some value of j, the terms which do make sense give the asymptotic behaviour near $r = 0$ (i.e., near $\delta = 0$) [2.5]. In what follows, we assume that the moments exist so far as we need them for computation.

Wherever the small parameter r appears explicitly in (6.2.15), we need to replace it by an expression in the small parameter δ. Additionally, we will expand $\hat{\varepsilon}$ in a series in δ, the form of which series depends on the strain curve f. To first determine how r depends upon δ, we use equation (6.2.4) and the equalities $U^2 = U_e^2(1+\delta) = J_e^{-1}(1+\delta)$ to write:

$$r\bar{J}(r) = J_e(1 - \delta + \delta^2 - \delta^3 + \cdots). \tag{6.2.16}$$

Recall that $\bar{J}(r)$ is the Laplace transform of $J(t)$. We transform equation (5.1.8) to obtain:

$$r\bar{J}(r) = J_e\left[1 - r\bar{\psi}(r)\right]. \tag{6.2.17}$$

For small r:

$$\begin{aligned}
\bar{\psi}(r) &= \int_0^\infty e^{-rt}\psi(t)\,dt \\
&= \int_0^\infty \left[1 - rt + \frac{r^2}{2!}t^2 - + \cdots\right]\psi(t)\,dt \\
&= \psi_0 - \psi_1 r + \frac{1}{2!}\psi_2 r^2 - + \cdots,
\end{aligned} \tag{6.2.18}$$

with the use of (6.2.13).

Using (6.2.18) in (6.2.17), we have:

$$r\bar{J}(r) = J_e\left[1 - \psi_0 r + \psi_1 r^2 - \frac{1}{2!}\psi_2 r^3 + - \cdots\right]. \tag{6.2.19}$$

Comparison of (6.2.19) with (6.2.16) suggests a power series expansion of r in δ:

$$r = \delta r_1 + \delta^2 r_2 + \delta^3 r_3 + \cdots. \tag{6.2.20}$$

When we use (6.2.20) in (6.2.19) and equate the result with (6.2.16), we find that the coefficients of the powers of δ in the expansion of r are:

$$\begin{aligned}
r_1 &= \frac{1}{\psi_0} \\
r_2 &= r_1\left(\psi_1 r_1^2 - 1\right) \\
r_3 &= r_1\left(-\frac{1}{2}\psi_2 r_1^3 + 2\psi_1 r_1 r_2 + 1\right),
\end{aligned} \tag{6.2.21}$$

and so on.

We insert (6.2.20) in (6.2.15), use $U^2 J_e = 1 + \delta$, and collect the powers of δ to write

$$\begin{aligned} U^2(\hat{J} \star \hat{\varepsilon}')(\theta) - \hat{\varepsilon}(\theta) &= \delta(\hat{\varepsilon}(\theta) - \hat{\varepsilon}'(\theta)) \\ &\quad + \delta^2 r_1^2 \psi_1 \Big(\hat{\varepsilon}''(\theta) - \hat{\varepsilon}'(\theta)\Big) \\ &\quad + O(\delta^3 \hat{\varepsilon}(\theta)), \end{aligned} \tag{6.2.22}$$

where we have used $r_1 \psi_0 = 1$ in the $O(\delta)$ term and $1 + r_2 \psi_0 = r_1^2 \psi_1$ in the $O(\delta^2)$ term, in view of equation (6.2.21).

So far, we have yet to expand $\hat{\varepsilon}$ in a perturbation series in δ. For the first case we consider, the strain curve is specified only as in (2.3.18), and the expansion of $\hat{\varepsilon}$ is crude:

$$\hat{\varepsilon}(\theta) = \delta^a \hat{\varepsilon}_1(\theta) + o(\delta^a), \tag{6.2.23}$$

where the exponent a is to be determined and $\hat{\varepsilon}_1(\theta)$ and its derivatives are $O(1)$. We insert (6.2.23) in (6.2.22) and obtain:

$$U^2(\hat{J} \star \hat{\varepsilon}')(\theta) - \hat{\varepsilon}(\theta) = \delta^{a+1}(\hat{\varepsilon}_1(\theta) - \hat{\varepsilon}_1'(\theta)) + o(\delta^{a+1}). \tag{6.2.24}$$

From (2.3.18) we have:

$$f(\hat{\varepsilon}) - \hat{\varepsilon} = k\hat{\varepsilon}^\gamma + o(\hat{\varepsilon}^\gamma). \tag{6.2.25}$$

Using (6.2.23) in (6.2.25), we obtain:

$$f\big(\hat{\varepsilon}(\theta)\big) - \hat{\varepsilon}(\theta) = \delta^{\gamma a} k \big(\hat{\varepsilon}_1(\theta)\big)^\gamma + o(\delta^{\gamma a}). \tag{6.2.26}$$

When we put (6.2.24) and (6.2.26) in the governing equation (6.2.8), the lowest order terms will enter with the same order of δ provided

$$a = \frac{1}{\gamma - 1}. \tag{6.2.27}$$

Thus,

$$\varepsilon(t) = \hat{\varepsilon}(\theta) = \delta^{\frac{1}{\gamma-1}} \hat{\varepsilon}_1(\theta) + o(\delta^{\frac{1}{\gamma-1}}), \tag{6.2.28}$$

and $\hat{\varepsilon}_1$ is determined to within an integration constant from:

$$\hat{\varepsilon}_1' - \hat{\varepsilon}_1 + k\hat{\varepsilon}_1^{\gamma} = 0. \tag{6.2.29}$$

To integrate (6.2.29), we note that it is separable,

$$\frac{1}{1 - k\hat{\varepsilon}_1^{\gamma-1}} \cdot \frac{d\hat{\varepsilon}_1}{\hat{\varepsilon}_1} = d\theta \tag{6.2.30}$$

and may be put in the form

$$\frac{d\eta}{1 - ke^{\eta/a}} = d\theta, \tag{6.2.31}$$

where a is given in (6.2.27) and

$$\eta = \ln \hat{\varepsilon}_1. \tag{6.2.32}$$

Upon integration of (6.2.31), we have:

$$ke^{\eta/a} = \frac{e^{(\theta-\theta_0)/a}}{1 + e^{(\theta-\theta_0)/a}}, \tag{6.2.33}$$

where θ_0 is the integration constant. In this problem, the time origin is arbitrary. We choose to set this origin with $\theta_0 = 0$. Using (6.2.27) and (6.2.32) to write the result in terms of $\hat{\varepsilon}_1$, we have:

$$\hat{\varepsilon}_1(\theta) = \left\{ \frac{1}{k} \cdot \frac{e^{(\gamma-1)\theta}}{1 + e^{(\gamma-1)\theta}} \right\}^{\frac{1}{\gamma-1}}. \tag{6.2.34}$$

This may be written in terms of the hyperbolic tangent as:

$$\hat{\varepsilon}_1(\theta) = \left\{ \frac{1}{2k} \left[1 + \tanh\left(\frac{\gamma-1}{2}\theta \right) \right] \right\}^{\frac{1}{\gamma-1}}. \tag{6.2.35}$$

With the use of (6.2.5), we observe from (6.2.34) that $\hat{\varepsilon}_1$ has the asymptotic form required by (6.2.3), to within the arbitrary time shift implicit in the multiplicative constant $k^{-1/(\gamma-1)}$.

We now consider this problem in more detail for strain curves amenable to a power series expansion:

$$f(\epsilon) = \varepsilon + \alpha\varepsilon^2 + \beta\varepsilon^3 + \cdots. \tag{6.2.36}$$

In this case, it is reasonable to assume, at the outset, that

$$\hat{\varepsilon}(\theta) = \delta\hat{\varepsilon}_1(\theta) + \delta^2\hat{\varepsilon}_2(\theta) + O(\delta^3), \tag{6.2.37}$$

where the functions $\hat{\varepsilon}_1(\theta)$ and $\hat{\varepsilon}_2(\theta)$ and their derivatives are $O(1)$ with respect to δ. We use (6.2.37) in (6.2.36) and obtain:

$$f(\hat{\varepsilon}) - \hat{\varepsilon} = \delta^2\alpha\hat{\varepsilon}_1^2 + \delta^3(2\alpha\hat{\varepsilon}_1\hat{\varepsilon}_2 + \beta\hat{\varepsilon}_1^3) + O(\delta^4). \tag{6.2.38}$$

Similarly, from (6.2.37) and (6.2.22) we have:

$$\begin{aligned} U^2(\hat{J} \star \hat{\varepsilon}') - \hat{\varepsilon} = {} & \delta^2(\hat{\varepsilon}_1 - \hat{\varepsilon}_1') \\ & + \delta^3\Big(\hat{\varepsilon}_2 - \hat{\varepsilon}_2' + r_1^2\psi_1(\hat{\varepsilon}_1'' - \hat{\varepsilon}_1')\Big) \\ & + O(\delta^4). \end{aligned} \tag{6.2.39}$$

We equate (6.2.38) and (6.2.39) according to (6.2.8) and obtain equations which $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ must satisfy. From the $O(\delta^2)$ terms, we get equation (6.2.29) for this case:

$$\hat{\varepsilon}_1' - \hat{\varepsilon}_1 + \alpha\hat{\varepsilon}_1^2 = 0. \tag{6.2 40}$$

The $O(\delta^3)$ terms produce an inhomogeneous differential equation for $\hat{\varepsilon}_2$ with variable coefficients which are continuous functions of θ:

$$\hat{\varepsilon}_2'(\theta) - a(\theta)\hat{\varepsilon}_2(\theta) = b(\theta), \tag{6.2.41}$$

where

$$\begin{aligned} a(\theta) &= \big(1 - 2\alpha\hat{\varepsilon}_1(\theta)\big) \\ b(\theta) &= r_1^2\psi_1(\hat{\varepsilon}_1''(\theta) - \hat{\varepsilon}_1'(\theta)) - \beta\hat{\varepsilon}_1^3(\theta). \end{aligned} \tag{6.2.42}$$

We obtain $\hat{\varepsilon}_1$ immediately from equation (6.2.35):

$$\hat{\varepsilon}_1(\theta) = \frac{1}{2\alpha}\Big[1 + \tanh(\theta/2)\Big]. \tag{6.2.43}$$

The solution for $\hat{\varepsilon}_2$ is the sum of a particular integral and the homogeneous solution:

$$\hat{\varepsilon}_2(\theta) = \hat{\varepsilon}_2^p(\theta) + \hat{\varepsilon}_2^h(\theta). \tag{6.2.44}$$

The solution for the homogeneous equation is in general:

$$\hat{\varepsilon}_2^h(\theta) = C \exp\left\{ \int^{\theta} a(\xi)\, d\xi \right\}, \tag{6.2.45}$$

where C is the integration constant. With $a(\theta)$ and $\hat{\varepsilon}_1(\theta)$ given by (6.2.42) and (6.2.43), the integration is for $\hat{\varepsilon}_2^h$ is straightforward. The result is:

$$\hat{\varepsilon}_2^h(\theta) = C \cosh^{-2}(\theta/2). \tag{6.2.46}$$

For $\hat{\varepsilon}_2^p(\theta)$, we have in general:

$$\hat{\varepsilon}_2^p(\theta) = \int^{\theta} \exp\left\{ \int_{\eta}^{\theta} a(\xi)\, d\xi \right\} b(\eta)\, d\eta. \tag{6.2.47}$$

Using $\hat{\varepsilon}_1(\theta)$ in equations (6.2.42) for $a(\theta)$ and $b(\theta)$, we have after much simplification, the intermediate result:

$$\hat{\varepsilon}_2^p(\theta) = \cosh^{-2}(\theta/2) \sum_{n=0}^{3} c_n \int^{\theta/2} \sinh^n(\xi) \cosh^{2-n}(\xi) d\xi, \tag{6.2.48}$$

where

$$\begin{aligned} c_0 &= -\frac{1}{2\alpha}\left(\frac{\psi_1}{\psi_0^2} + \frac{\beta}{2\alpha^3}\right), & c_2 &= \frac{1}{2\alpha}\left(\frac{\psi_1}{\psi_0^2} - \frac{3\beta}{2\alpha^3}\right), \\ c_1 &= -\frac{1}{2\alpha}\left(\frac{\psi_1}{\psi_0^2} + \frac{3\beta}{2\alpha^3}\right), & c_3 &= \frac{1}{2\alpha}\left(\frac{\psi_1}{\psi_0^2} - \frac{\beta}{2\alpha^3}\right), \end{aligned} \tag{6.2.49}$$

for which we have used (6.2.21) to write the c_n in terms of the compliance moments, ψ_0 and ψ_1, defined in (6.2.13). Again, the integrations in (6.2.48) are straightforward. After integrating and simplifying the result for $\hat{\varepsilon}_2^p$, we add it to $\hat{\varepsilon}_2^h$, according to (6.2.44). The result for $\hat{\varepsilon}_2(\theta)$ is:

$$\begin{aligned} \hat{\varepsilon}_2(\theta) = &-\frac{\beta}{2\alpha^3}\Big[\tanh(\theta/2) + \tanh^2(\theta/2)\Big] \\ &+ \left\{\frac{1}{\cosh^2(\theta/2)}\right\}\left\{C + \frac{1}{2\alpha}\left(\frac{\beta}{2\alpha^3} - \frac{\psi_1}{\psi_0^2}\right)\Big((\theta/2) + \ln\big[\cosh(\theta/2)\big]\Big)\right\}. \end{aligned} \tag{6.2.50}$$

To verify that $\hat{\varepsilon}_2(\theta)$ has the correct asymptotic property for $\theta \to -\infty$, we use the following identities:

$$\begin{aligned} \cosh x &= \frac{e^{x} + e^{-x}}{2} \\ \tanh x &= \frac{2e^{x}}{e^{x} + e^{-x}} - 1. \end{aligned} \tag{6.2.51}$$

For large negative x:

$$\left.\begin{aligned} \cosh x &\sim \frac{e^{-x}}{2}, \\ \tanh x &\sim 2e^{2x} - 1, \end{aligned}\right\} \qquad x \to -\infty. \tag{6.2.52}$$

Thus,

$$\left.\begin{aligned} \tanh(\theta/2) + \tanh^2(\theta/2) &\sim -2e^{\theta}, \\ \cosh^{-2}(\theta/2) &\sim 4e^{\theta} \\ \ln\left[\cosh(\theta/2)\right] &\sim -(\theta/2) - \ln 2, \end{aligned}\right\} \qquad \theta \to -\infty. \tag{6.2.53}$$

Using (6.2.52) and (6.2.53) in (6.2.50), we obtain:

$$\hat{\varepsilon}_2(\theta) \sim \left\{ \frac{\beta}{\alpha^3} + 4\left[C - \frac{1}{2\alpha}\left(\frac{\beta}{2\alpha^3} - \frac{\psi_1}{\psi_0^2}\right) \ln 2\right]\right\} e^{\theta}, \qquad \theta \to -\infty. \tag{6.2.54}$$

As in the solution for $\hat{\varepsilon}_1(\theta)$, the integration constant C serves to define the arbitrary time shift of the solution.

With $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ given in (6.2.43) and (6.2.50), our solution for $\hat{\varepsilon}(\theta)$ is complete, so far as we have expanded it in equation (6.2.37). Since $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ are bounded in θ, the expansion is uniformly asymptotic. To go on to higher orders in the solution, one merely continues the expansion of (6.2.37) with terms $\delta^3 \hat{\varepsilon}_3(\theta) + O(\delta^4)$ and obtains an equation for $\hat{\varepsilon}_3(\theta)$ in terms of $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$.

As a final remark on the solution we have obtained, we show that it produces the correct equilibrium value, to within $O(\delta^3)$. From (6.2.43) and (6.2.50), respectively, we have:

$$\begin{aligned} \lim_{\theta \to +\infty} \hat{\varepsilon}_1(\theta) &= +\frac{1}{\alpha} \\ \lim_{\theta \to +\infty} \hat{\varepsilon}_2(\theta) &= -\frac{\beta}{\alpha^3}. \end{aligned} \tag{6.2.55}$$

Thus, the equilibrium value for $\hat{\varepsilon}(\theta)$ from (6.2.37) is:

$$\hat{\varepsilon}_e = \frac{\delta}{\alpha} - \frac{\beta \delta^2}{\alpha^3} + O(\delta^3). \tag{6.2.56}$$

We recall from section 3.3, equation (3.3.2), that the equilibrium value satisfies:

$$f(\hat{\varepsilon}_e) = U^2 J_e \hat{\varepsilon}_e. \tag{6.2.57}$$

Using $U^2 J_e = 1 + \delta$ and the expansion of f in (6.2.36), we have from (6.2.57):

$$\alpha \hat{\varepsilon}_e + \beta \hat{\varepsilon}_e^2 + O(\hat{\varepsilon}_e^3) = \delta. \tag{6.2.58}$$

When we solve (6.2.58) for $\hat{\varepsilon}_e$ in terms of δ, we obtain equation (6.2.56), the required result.

6.3 Power Law Materials with Quadratic Nonlinearity.

We now obtain steady wave solutions at all wavespeeds below that of the acceleration wave for materials having a purely quadratic nonlinearity in the strain curve:

$$f(\varepsilon) = \varepsilon + \alpha \varepsilon^2, \qquad \alpha > 0. \tag{6.3.1}$$

Our solutions apply for two different classes of material compliance models. The first class is characterized by compliances J which have a non-zero initial jump, J_o, followed by a power law *normalized* compliance (§§4.6):

$$\begin{aligned} J(t) &= J_o[H(t) + CP(t)] \\ P(t) &= \frac{t^p}{p!}, \qquad p \in (0, 1], \end{aligned} \tag{6.3.2}$$

where $p!$ is defined in equation (2.4.11) and C is a positive constant. Another class of material models is defined by a power law compliance with no initial jump:

$$J(t) = \tilde{C} P(t), \qquad p \in (0, 1], \tag{6.3.3}$$

where $\tilde{C}$ is a positive constant. In such a model, p is the log-log slope of the time dependence of J and is fairly constant over broad ranges of t in real materials (see, e.g., Ferry [2.3]). In either model, $p = 0$ corresponds to purely elastic materials. Additionally, neither compliance model has a finite equilibrium value; the equilibrium wavespeed $U_e^2 = 1/J_e$ is therefore zero. Steady waves can exist in these materials for all non-zero values of wavespeed. Since the power law material in equation (6.3.3) has no initial jump, it cannot support steady shock or acceleration wave solutions; the acceleration wavespeed $U_o^2 = 1/J_o$ is infinite for $J_o = 0$. For such materials, the steady wave solutions produced in this section apply for all finite wavespeeds $U \in (0, \infty)$.

For either compliance, we reduce the problem to that of solving the same nonlinear integral equation with a singular kernel. For the normalized power law model (6.3.2) with the strain curve of equation (6.3.1), the governing equation (3.1.10) yields:

$$(1-\lambda)\varepsilon + \alpha\varepsilon^2 = \lambda C(P' \star \varepsilon)(t), \tag{6.3.4}$$

for which we recall the definition of λ given in equation (5.3.3):

$$\lambda = U^2 J_o = (U/U_o)^2. \tag{6.3.5}$$

For the wavespeeds of this section, we have $\lambda < 1$. We introduce the scaled strain

$$u(t) = \frac{\alpha}{1-\lambda}\varepsilon(t) \tag{6.3.6}$$

and obtain from (6.3.4):

$$u(t) + u^2(t) = \frac{\lambda C}{1-\lambda}(P' \star u)(t). \tag{6.3.7}$$

The positive constant multiplying the convolution may be absorbed by introducing the scaled non-dimensional time

$$\eta = \left(\frac{\lambda C}{1-\lambda}\right)^{1/p} t. \tag{6.3.8}$$

We then have

$$\overline{u}(\eta) + \overline{u}^2(\eta) = (P' \star \overline{u})(\eta), \tag{6.3.9}$$

where

$$\overline{u}(\eta) = u(t). \tag{6.3.10}$$

Similarly, for a pure power law compliance defined in (6.3.3), equation (6.3.9) obtains when u and η are defined as

$$u(t) = \alpha\varepsilon(t) \qquad \text{and} \qquad \eta = (U^2\tilde{C})^{1/p}\, t. \tag{6.3.11}$$

For both classes of materials, equation (6.3.9) is the nonlinear integral equation to be solved for $\overline{u}$. The kernel $P'(\eta)$ is singular for $p \in (0,1)$. If $p = 1$, the kernel is the Heaviside step function, with which convolution is a pure integration. In this case, we have

$$\overline{u}(\eta) + \overline{u}^2(\eta) = \int_{-\infty}^{\eta} \overline{u}(s)\, ds. \tag{6.3.12}$$

For any differentiable nonlinear function $f(u)$, the integral equation

$$f(u) = \int_{-\infty}^{t} u(\tau)\, d\tau \tag{6.3.13}$$

may be differentiated to obtain

$$f'(u)d(\ln u) = d\tau. \tag{6.3.14}$$

Integration of (6.3.14) gives u implicitly:

$$\int^{u} f'(s)\,(ds/s) = t - t_o, \tag{6.3.15}$$

for an arbitrary shift t_o. For our quadratically nonlinear problem with $p = 1$, the exact solution from (6.3.12) is therefore:

$$\ln \overline{u}(\eta) + 2\overline{u}(\eta) = \eta - \eta_o. \tag{6.3.16}$$

For values of $p \in (0,1)$, the singular kernel in the convolution complicates the problem. Our solution combines a series representation for the exponentially small "toe" of the wave, valid until some $\eta_T \in (-\infty, 0)$, with an iterative solution constructed on the interval $[\eta_T, \eta]$, for any $\eta > \eta_T$. We first discuss the series solution. In the proof of Theorem 4.5.1, the existence theorem for these wavespeeds, we used bounds analogous to

$$e^{\eta} - C' e^{2\eta} < \overline{u}(\eta) < e^{\eta}, \tag{6.3.17}$$

for η sufficiently small with a positive constant C'. This suggests we try an alternating series solution in powers of e^{η}:

$$\overline{u}(\eta) = \sum_{k=0}^{\infty} c_k e^{(k+1)\eta}. \tag{6.3.18}$$

We assume at this point that this series is absolutely convergent for η small enough so that we may square it for use in (6.3.9). We will prove that this is so once the *coefficients* $\{c_i\}_{i=0}^{\infty}$ *are determined*. We have

$$\overline{u}^2(\eta) = \sum_{k=0}^{\infty} (c \star c)_k e^{(k+2)\eta}, \tag{6.3.19}$$

where $(c \star c)_k$ represents the discrete convolution of coefficients:

$$(c \star c)_k = \sum_{j=0}^{k} c_{k-j} c_j. \tag{6.3.20}$$

We need the convoluton of P' with an arbitrary exponential:

$$\begin{aligned}(P' \star e^{k\eta})(\eta) &= \frac{1}{p!} \int_{-\infty}^{\eta} p(\eta - s)^{p-1} e^{ks}\, ds \\ &= k^{-p} e^{k\eta},\end{aligned} \tag{6.3.21}$$

with the use of (2.4.11). We use equations (6.3.18), (6.3.19), and (6.3.21) in (6.3.9) and obtain:

$$c_0 + \sum_{k=1}^{\infty} c_k e^{k\eta} + \sum_{k=1}^{\infty} (c \star c)_{k-1} e^{k\eta} = c_0 K_0 + \sum_{k=1}^{\infty} c_k K_k e^{k\eta}, \tag{6.3.22}$$

where

$$K_k = \frac{1}{(k+1)^p}, \qquad k = 0, 1, 2, \dots . \tag{6.3.23}$$

With the observation that $K_0 = 1$, we see that c_0 is arbitrary; we choose $c_0 = 1$ to obtain the solution which, according to (6.3.18), is asymptotic to e^η for $\eta \to -\infty$. Since $\{e^{k\eta}\}_{k=1}^{\infty}$ is a linearly independent set, the coefficients in the series of (6.3.18) are determined from the recursion relation:

$$\begin{aligned} c_0 &= 1 \\ c_k &= \frac{-(c \star c)_{k-1}}{1 - K_k}, \qquad k = 1, 2, 3, \dots . \end{aligned} \tag{6.3.24}$$

We note that $(c \star c)_0 = c_0^2 = 1$. Furthermore, $K_k \in (0, 1)$ for $k \geq 1$. Thus, $c_1 < 0$. It follows by induction that the coefficients c_k alternate in sign, since the summation (the discrete convolution) involves terms which are either all negative or all positive.

Given the series for $\overline{u}$ defined by (6.3.18) and (6.3.24), we must show that it converges absolutely on a non-empty interval $(-\infty, \eta_T]$. Viewing (6.3.18) as a power series in e^η, we are, in effect, looking for a non-zero lower bound on the radius of convergence of the series

$$S_1(z) = \sum_{k=0}^{\infty} |c_k| z^k . \tag{6.3.25}$$

It is difficult to directly estimate the growth of the coefficients in this series. We note for later comparison that the coefficients obey

$$|c_k| > (c \star c)_{k-1}, \qquad k = 1, 2, \dots . \tag{6.3.26}$$

We instead consider the series

$$S_2(z) = \sum_{k=0}^{\infty} b_k z^k , \tag{6.3.27}$$

with strictly positive coefficients defined by

$$b_k = (-1)^k c_k (1 - K_1)^k , \qquad k = 0, 1, 2, \dots . \tag{6.3.28}$$

If S_2 converges for $|z| < R$, then S_1 does so for $|z| < (1 - K_1)R$. Consequently, the series in (6.3.18) for $\overline{u}$ converges absolutely for $\eta < \ln\left[(1 - K_1)R\right]$. To show the convergence of S_2, we first determine the recursion relation for its coefficients. We will then show that these coefficients are dominated by those of a power series with a non-zero radius of convergence. Using equations (6.3.24), (6.3.19), and (6.3.28), we obtain the recursion relation:

$$\begin{aligned} b_0 &= 1 \\ b_k &= \frac{1 - K_1}{1 - K_k}(b \star b)_{k-1}, \qquad k = 1, 2, \ldots . \end{aligned} \tag{6.3.29}$$

Observe that $K_k < K_1$ for all $k \geq 2$. Therefore,

$$b_k \leq (b \star b)_{k-1}, \qquad k = 1, 2, \ldots, \tag{6.3.30}$$

with equality only at $k = 1$. We have transformed series S_1 whose coefficients are bounded below by their convolution, equation (6.3.26), into series S_2 for which the coefficient convolution provides an upper bound. Consider now the sequence $\{d_k\}_{k=0}^{\infty}$ of positive numbers defined by:

$$\begin{aligned} d_0 &= 1 \\ d_k &= (d \star d)_{k-1}. \end{aligned} \tag{6.3.31}$$

Since $b_0 = d_0 = 1$, it follows from (6.3.30) and (6.3.31) that

$$b_k \leq d_k, \qquad k = 0, 1, 2, \ldots, \tag{6.3.32}$$

with equality only at $k = 0, 1$. The sequence $\{d_k\}_{k=0}^{\infty}$ *dominates* the sequence $\{b_k\}_{k=0}^{\infty}$. We now exhibit a function which generates a power series having a non-zero radius of convergence with coefficients which obey the recursion relation (6.3.31). It is

$$g(z) = \frac{1}{2z}\left(1 - \sqrt{1 - 4z}\right), \tag{6.3.33}$$

with the branch of the square root which is positive when its argument is real and positive. For $|z| < 1/4$, we may write g as a convergent power series:

$$g(z) = \sum_{k=0}^{\infty} g_k z^k, \tag{6.3.34}$$

where the sequence of coefficients $\{g_k\}_{k=0}^{\infty}$ can be determined, for example, by expanding the square root in a binomial series. In particular,

$$g_0 = 1$$

and

$$g_k > 0 \quad \text{for all } k. \tag{6.3.35}$$

The series is absolutely convergent and we may square it:

$$g^2(z) = \sum_{k=0}^{\infty} (g \star g)_k z^k. \tag{6.3.36}$$

From (6.3.33), we eliminate the radical to obtain

$$z(g - 1) = z^2 g^2. \tag{6.3.37}$$

Using equations (6.3.34) and (6.3.36) in (6.3.37), we have

$$\sum_{k=1}^{\infty} g_k z^{k+1} = \sum_{k=0}^{\infty} (g \star g)_k z^{k+2}, \tag{6.3.38}$$

where we have used $g_0 z^0 = 1$. Upon adjustment of the indicies on the right summation, we see that

$$g_k = (g \star g)_{k-1}, \qquad k = 1, 2, \ldots, \tag{6.3.39}$$

in view of the linear independence of $\{z^k\}_{k=1}^{\infty}$. Since $g_0 = 1$, the coefficient sequence $\{g_k\}_{k=0}^{\infty}$ is identical to the sequence $\{d_k\}_{k=0}^{\infty}$. Thus, the convergence for $|z| < 1/4$ of the series for g which dominates S_2 implies the convergence of S_2 in the same circle. Hence, for any η_T which satisfies

$$\eta_T < \ln\left[(1 - K_1)/4\right], \tag{6.3.40}$$

the series for $\overline{u}(\eta)$ in (6.3.18) converges absolutely, at least for all $\eta \leq \eta_T$. Since $1 - K_1 < 1$, we have $\eta_T < 0$. We note that the series S_1 diverges for $|z| > 1/4$. For, equations (6.3.26) and (6.3.39) with $g_0 = 1$ imply that the coefficients of $S_1(z)$ dominate those of the series for $g(z)$, which series for g diverges outside the circle $|z| = 1/4$. While the series for $\overline{u}$ **may** converge absolutely for some η larger than η_T, it diverges absolutely for $\eta > \ln(1/4)$, for all values of p.

We have thus far solved the problem on the half-line $\eta \leq \eta_T$, where $\overline{u}(\eta)$ is given exactly by the alternating series defined by equations (6.3.18) and (6.3.24). We construct the solution for $\eta > \eta_T$ by iteration of monotone sequences of monotone functions, much as we did for the earlier constructions of shocks and acceleration waves. For the current problem, however, each iterate involves, for $\eta > \eta_T$, a contribution from the series solution for $\eta \leq \eta_T$. The iteration scheme is:

$$\overline{u}_{k+1}(\eta) = F\Big(\big(P' \star \overline{u}_k\big)(\eta)\Big), \tag{6.3.41}$$

where we now denote by F the inverse of $\overline{u} + \overline{u}^2$. We first show that an increasing (lower bound) sequence is started by

$$\overline{u}_o(\eta) = \begin{cases} \overline{u}(\eta), & \eta \leq \eta_T; \\ \overline{u}(\eta_T), & \eta \geq \eta_T; \end{cases} \tag{6.3.42}$$

where $\overline{u}$ is the known series solution for $\eta \leq \eta_T$. Recall that $\overline{u}$ is monotone, according to Theorem 4.5.1. For $\eta > \eta_T$,

$$\begin{aligned} (P' \star \overline{u}_o)(\eta) &= \int_{-\infty}^{\eta} P(\eta - s)\, d\overline{u}_o(s) \\ &= \int_{-\infty}^{\eta_T} P(\eta - s)\, d\overline{u}_o(s) \\ &> (P' \star \overline{u})(\eta_T), \end{aligned} \tag{6.3.43}$$

where we have used the increasing nature of $\overline{u}_o$ and P along with the fact that $d\overline{u}_o = 0$ for $\eta > \eta_T$. Since F, too, is increasing, we have

$$\overline{u}_1(\eta) = F\Big(\big(P' \star \overline{u}_o\big)(\eta)\Big) > F\Big(\big(P' \star \overline{u}_o\big)(\eta_T)\Big) = \overline{u}(\eta_T) = \overline{u}_o(\eta), \quad \eta > \eta_T \tag{6.3.44}$$

and the assertion is proven. Furthermore, $\overline{u}_1$ is bounded above by e^{η}, in view of Lemma 4.5.1. Theorem 4.5.1 ensures that iteration of $\overline{u}_o$ converges to the solution $\overline{u}$.

Similarly, a decreasing (upper bound) sequence is started by the discontinuous starter defined by:

$$\overline{u}_o(\eta) = \begin{cases} \overline{u}(\eta), & \eta \leq \eta_T; \\ e^{\eta}, & \eta > \eta_T. \end{cases} \tag{6.3.45}$$

For $\eta > \eta_T$,

$$\Big((P' \star \overline{u}_o)(\eta)\Big) < P' \star e^{\eta} = e^{\eta}, \tag{6.3.46}$$

according to Lemma 4.5.1, since $\overline{u}(\eta) < e^{\eta}$. Thus,

$$\overline{u}_1(\eta) < F(e^{\eta}) < e^{\eta} = \overline{u}_o(\eta), \qquad \eta > \eta_T. \tag{6.3.47}$$

All iterates in the resulting upper bound sequence are bounded below by the lower bound starter of equation (6.3.42), since $\overline{u}_1(\eta)$ is so bounded, in view of Lemma 4.3.4. Accordingly, the upper bound sequence converges to the solution $\overline{u}$. We remark that the discontinuity in the upper bound starter is not present in the succeeding iterates, since P is continuous.

We again resort to a simple numerical scheme to generate graphs of approximations to the upper and lower bound sequences discussed above. The results we present are **not** for solutions asymptotic to e^{η} for $\eta \to -\infty$. Rather, we consider the shifted solutions

$$\tilde{u}(\eta) = \overline{u}(\eta - |\eta_T|), \tag{6.3.48}$$

so that the series solution for $\tilde{u}$ is **always** used for $\eta \leq 0$. We have

$$\tilde{u}(\eta) = \sum_{k=0}^{\infty} \tilde{c}_k e^{(k+1)\eta}, \qquad \eta \leq 0, \tag{6.3.49}$$

where

$$\tilde{c}_k = c_k e^{-(k+1)|\eta_T|}, \tag{6.3.50}$$

and η_T is chosen according to (6.3.40). Any iterate in the sequence of bounding functions has the form:

$$\varphi(\eta) = \begin{cases} \tilde{u}(\eta), & \eta \leq 0; \\ \varphi(\eta), & \eta > 0. \end{cases} \tag{6.3.51}$$

The convolution with P' is:

$$\begin{aligned} (P' \star \varphi)(\eta) &= \int_0^\infty \varphi(\eta - s) P'(s)\, ds \\ &= \int_0^\eta \varphi(\eta - s) P'(s)\, ds + \int_\eta^\infty \tilde{u}(\eta - s) P'(s)\, ds. \end{aligned} \tag{6.3.52}$$

From equations (6.3.2) and (6.3.49), we see that the last integral in (6.3.52) is a sum of integrals of the form:

$$I(\eta) = \frac{1}{p!} \int_\eta^\infty e^{r(\eta - s)} p s^{p-1}\, ds. \tag{6.3.53}$$

With the change of variables $rs \to \xi$, we have

$$I(\eta) = \frac{r^{-p} e^{r\eta}}{(p-1)!} \int_{r\eta}^\infty e^{-\xi} \xi^{p-1}\, d\xi. \tag{6.3.54}$$

The integral in (6.3.54) is an *incomplete gamma function*. In the notation of Abramowitz and Stegun [5.1], we have

$$I(\eta) = r^{-p} e^{r\eta} \frac{\Gamma(p, r\eta)}{\Gamma(p)}, \tag{6.3.55}$$

for which the connection with our notation in (2.4.11) is provided by

$$p! = \Gamma(p+1). \tag{6.3.56}$$

Using equations (6.3.49), (6.3.53), and (6.3.55) in (6.3.52), the complete expression for the convolution for $\eta > 0$ is:

$$\begin{aligned} (P' \star \varphi)(\eta) = &\frac{1}{\Gamma(p)} \sum_{k=0}^\infty \tilde{c}_k \frac{e^{(k+1)\eta}}{(k+1)^p} \Gamma(p, (k+1)\eta) \\ &+ \frac{1}{p!} \int_0^\eta \varphi(\eta - s)\, dP(s). \end{aligned} \tag{6.3.57}$$

The alternating series in this result can be approximated arbitrarily closely by a finite number of terms with the magnitude of the error bounded by that of the first neglected term. To approximate the incomplete gamma function, we use the convergent series, continued fraction, and asymptotic series representations given, respectively, in equations (6.5.29), (6.5.31), and (6.5.32) of [5.1]. We approximate the convolution integral with the scheme presented in equations (5.5.15). (5.5.16), and (5.5.30) of this work.

Figure 6.1 illustrates truncated upper and lower bound sequences for a quadratically nonlinear material with $p = 1/4$. Here we chose $\eta_T = -4$ and we have plotted $\tilde{u}(\eta) = \overline{u}(\eta - 4)$ against η. It shows the lower bound starter of equation (6.3.42) and the first 19 iterates of the sequence it generates, along with the upper bound starter of equation (6.3.45) and the first 17 iterates of its sequence. For each approximate iterate, 100 points were computed and plotted for $\eta > 0$. For all figures in this section, 50 points are plotted for $\eta \leq 0$ where the alternating series solution is shown. Figures 6.3–6.6, discussed below, were prepared with 200 points computed and plotted for $\eta > 0$. As in the graphs of Chapter 5, the large space between the truncated sequences is the numerical approximation to the bound within which the solution is to be found.

We expect that a starting function which is similar to the expected form of the solution will yield better bounds upon iteration. We now present such a starter for an upper bound sequence. We define

$$B_1(\eta) = e^{\eta}, \tag{6.3.58}$$

and $B_2(\eta)$ as the solution to:

$$B_2^2(\eta) = (P' \star B_2)(\eta). \tag{6.3.59}$$

The solution for B_2 is

$$B_2(\eta) = \frac{p!}{(2p)!}(\eta - \eta_0)^p. \tag{6.3.60}$$

We show that the starting function defined by:

$$\overline{u}_o(\eta) = \begin{cases} B_1(\eta), & \eta \le \eta_1; \\ B_2(\eta), & \eta \ge \eta_1, \end{cases} \tag{6.3.61}$$

with

$$\eta_1 = \ln\left(\frac{p^p p!}{(2p)!}\right) \qquad \text{and} \qquad \eta_0 = \eta_1 - p, \tag{6.3.62}$$

starts a decreasing sequence upon iteration. These values of η_0 and η_1 produce a continuously differentiable starter $\overline{u}_o$. From the iteration scheme (6.3.41), we have

$$\overline{u}_1 + \overline{u}_1^2 = P' \star \overline{u}_o. \tag{6.3.63}$$

If, for all η, iteration produces $\overline{u}_1 + \overline{u}_1^2 < \overline{u}_o + \overline{u}_o^2$, then $\overline{u}_1 < \overline{u}_o$ and $\overline{u}_o$ starts a decreasing sequence. It is therefore sufficient to show that $P' \star \overline{u}_o < \overline{u}_o + \overline{u}_o^2$. Since $P' \star e^\eta = e^\eta$, the result is immediate for $\eta \le \eta_1$. For $\eta > \eta_1$,

$$\begin{aligned}(P' \star \overline{u}_o)(\eta) &= \int_{-\infty}^{\eta} \overline{u}_o(s)P'(\eta - s)\,ds \\ &= \int_{-\infty}^{\eta_1} B_1(s)P'(\eta - s)\,ds + \int_{\eta_1}^{\eta} B_2(s)P'(\eta - s)\,ds \\ &= \int_{-\infty}^{\eta_1} B_1(s)P'(\eta - s)\,ds + \int_{\eta_0}^{\eta} B_2(s)P'(\eta - s)\,ds \\ &\qquad - \int_{\eta_0}^{\eta_1} B_2(s)P'(\eta - s)\,ds.\end{aligned} \tag{6.3.64}$$

Since B_2 vanishes for $\eta < \eta_1$, we can extend the lower limit of integration in the last two integrals above to $-\infty$. Thus,

$$(P' \star \overline{u}_o)(\eta) = \int_{-\infty}^{\eta_1} (B_1 - B_2)(s)P'(\eta - s)\,ds + \int_{-\infty}^{\eta} B_2(s)P'(\eta - s)\,ds. \tag{6.3.65}$$

The last integral is $(P' \star B_2)(\eta) = B_2^2(\eta)$, according to equation (6.3.59). Therefore,

$$\begin{aligned}(P' \star \overline{u}_o)(\eta) &= B_2^2(\eta) + \int_{-\infty}^{\eta_1} (B_1 - B_2)(s)P'(\eta - s)\,ds \\ &< B_2^2(\eta) + \int_{-\infty}^{\eta_1} (B_1 - B_2)(s)P'(\eta_1 - s)\,ds,\end{aligned} \tag{6.3.66}$$

since $(B_1 - B_2)(\eta) > 0$ for all $\eta \neq \eta_1$ and $P'(\eta)$ is a decreasing function. The last integral in (6.3.66) is $P' \star (B_1 - B_2)(\eta_1) = B_1(\eta_1) - B_2^2(\eta_1)$. Therefore,

$$\begin{aligned}(P' \star \overline{u}_o)(\eta) &< B_2^2(\eta) + B_1(\eta_1) - B_2^2(\eta_1) \\ &< B_2^2(\eta) + B_1(\eta_1) \\ &= B_2^2(\eta) + B_2(\eta_1),\end{aligned} \tag{6.3.67}$$

since $\overline{u}_o$ is continuous at η_1. Now B_2 is increasing, Hence, we obtain:

$$\begin{aligned}(P' \star \overline{u}_o)(\eta) &< B_2^2(\eta) + B_2(\eta) \\ &= \overline{u}_o^2(\eta) + \overline{u}_o(\eta),\end{aligned} \tag{6.3.68}$$

which proves the assertion.

With this result, it is easy to show that an upper bound starter useful for our numerical work is provided by the discontinuous function defined as:

$$\overline{u}_o(\eta) = \begin{cases} \overline{u}(\eta), & \eta \leq \eta_T; \\ B_1(\eta), & \eta_T < \eta \leq \eta_1; \\ B_2(\eta), & \eta \geq \eta_1, \end{cases} \tag{6.3.69}$$

with η_0 and η_1 given in (6.3.62) and the series for $\overline{u}$ in (6.3.18). Figures 6.2–6.6 illustrate approximate bounds based upon this upper bound starter. In Figure 6.2 ($p = 1/4$) with 100 points computed and plotted for $\eta > 0$, we see that the early iterates are much better upper bounds than those started by the discontinuous exponential starter defined in equation (6.3.45) and illustrated in Figure 6.1.

Recall, we have the exact solution for $\overline{u}(\eta)$ for all η in the single case when $p = 1$. It is given in equation (6.3.16). In Figure 6.3, we compare the results of 20 iterations in both lower and upper bound sequences with the exact result to provide some verification of our numerical scheme. The approximate iterated solution and the exact solution are indistinguishable. Figures 6.4–6.6 illustrate our approximations to bounds on the solutions for $p = 1/2$, $1/4$, and $1/10$.

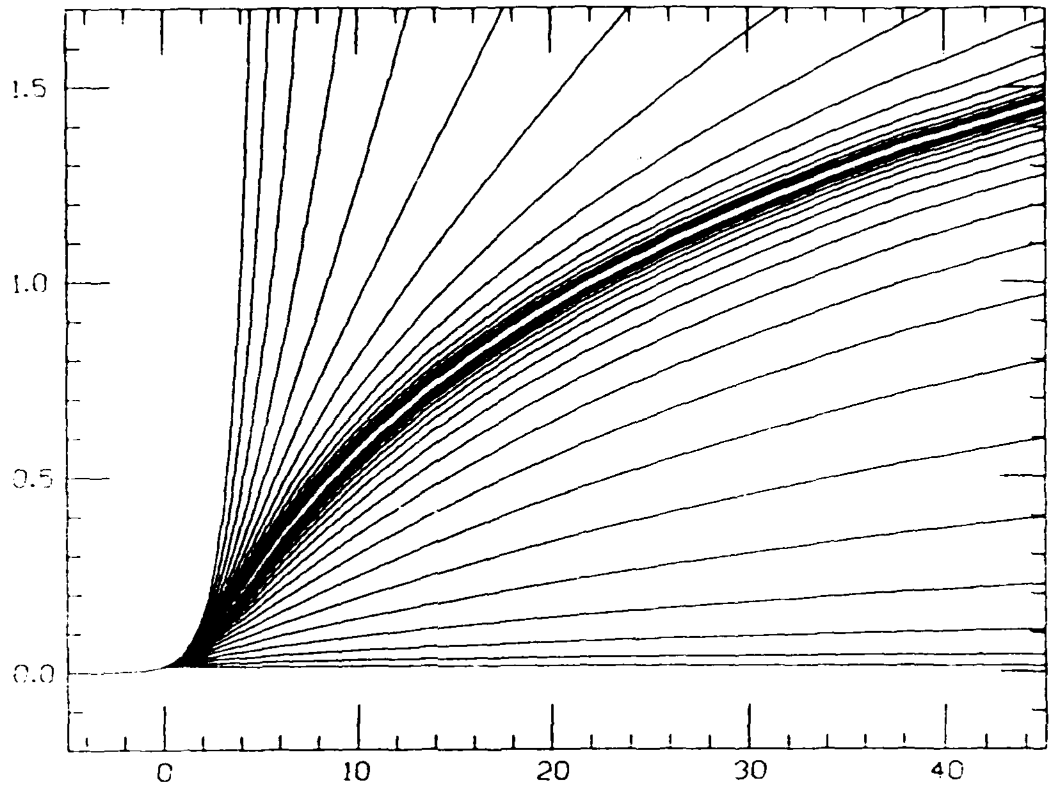

Figure 6.1. Approximations to bounding sequences for $p = 1/4$. The ordinate variable is $\hat{u}(\eta) = \overline{u}(\eta - 4)$. The abscissa represents the non-dimensional time η.

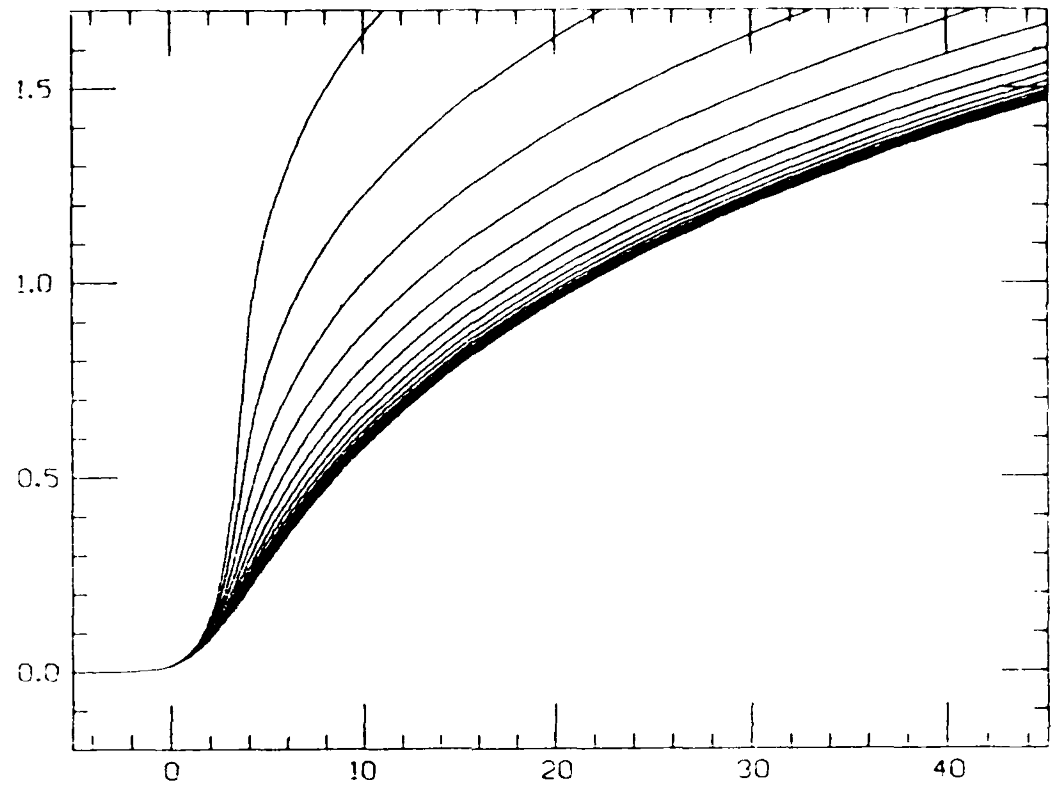

Figure 6.2. Upper bound starter of equation (6.3.69) and 17 iterates for $p = 1/4$. The ordinate variable is $\tilde{u}(\eta) = \overline{u}(\eta - 4)$. The abscissa represents the non-dimensional time η.

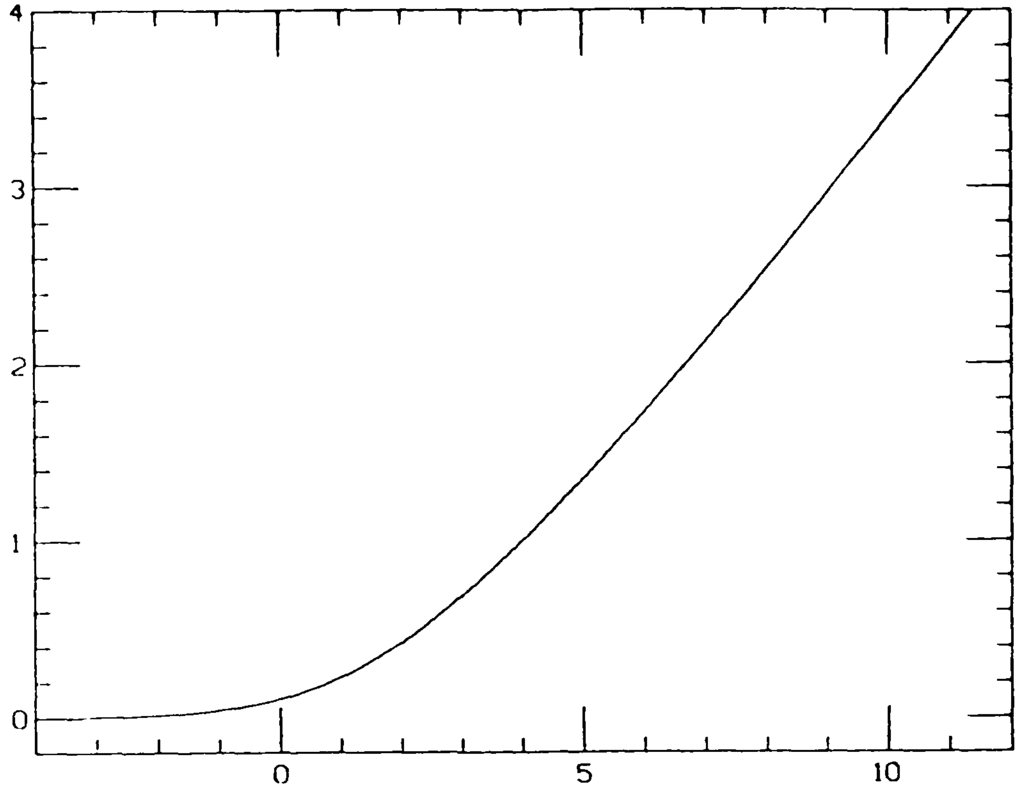

Figure 6.3. The exact solution for $p = 1$ and the superimposed 20-th iterates in the upper and lower bound sequences. The ordinate variable is $\bar{u}(\eta) = \widetilde{u}(\eta - 2)$. The abscissa represents the non-dimensional time η.

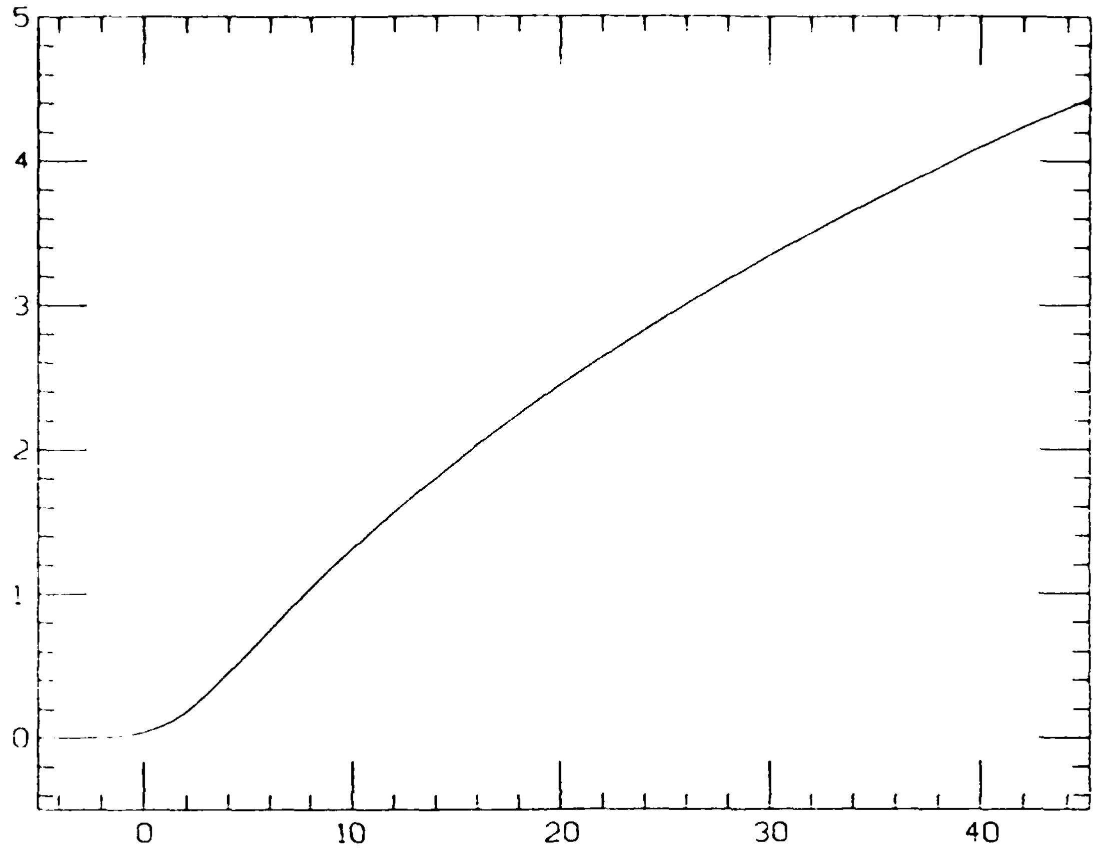

Figure 6.4. The superimposed 20-th upper bound and lower bound iterates for $p = 1/2$. The ordinate variable is $\tilde{u}(\eta) = \overline{u}(\eta - 3)$. The abscissa represents the non-dimensional time η.

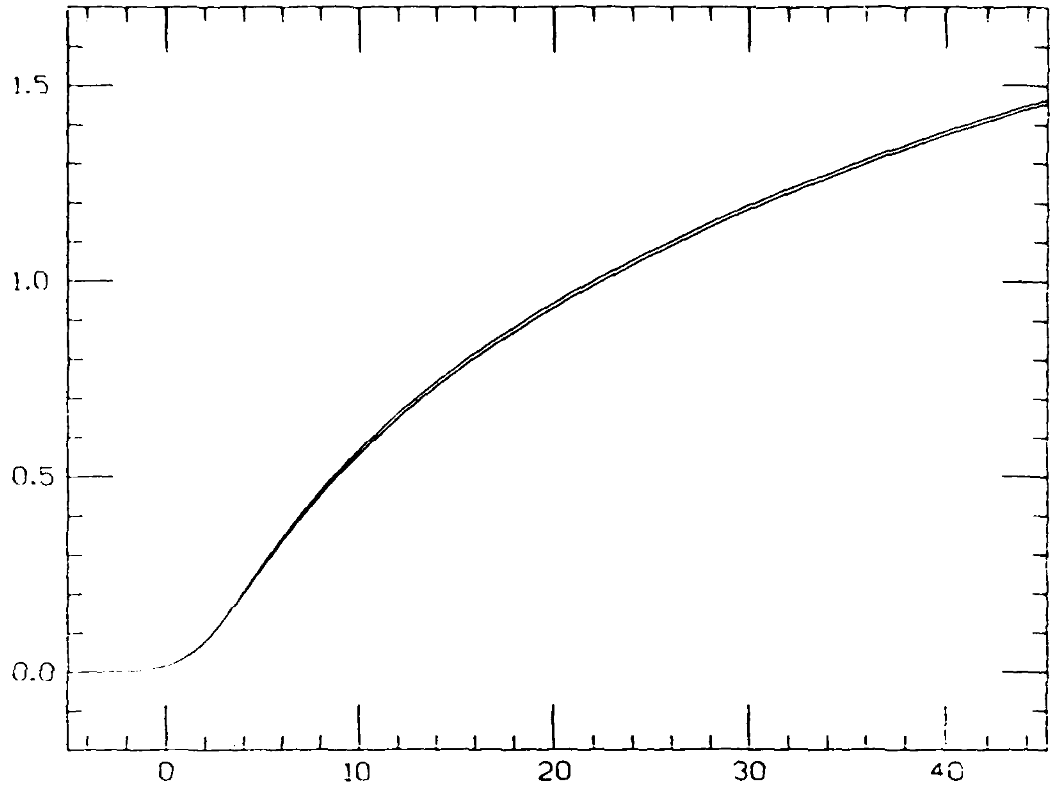

Figure 6.5. The 20-th upper bound and lower bound iterates for $p = 1/4$. The ordinate variable is $\tilde{u}(\eta) = \overline{u}(\eta - 4)$. The abscissa represents the non-dimensional time η.

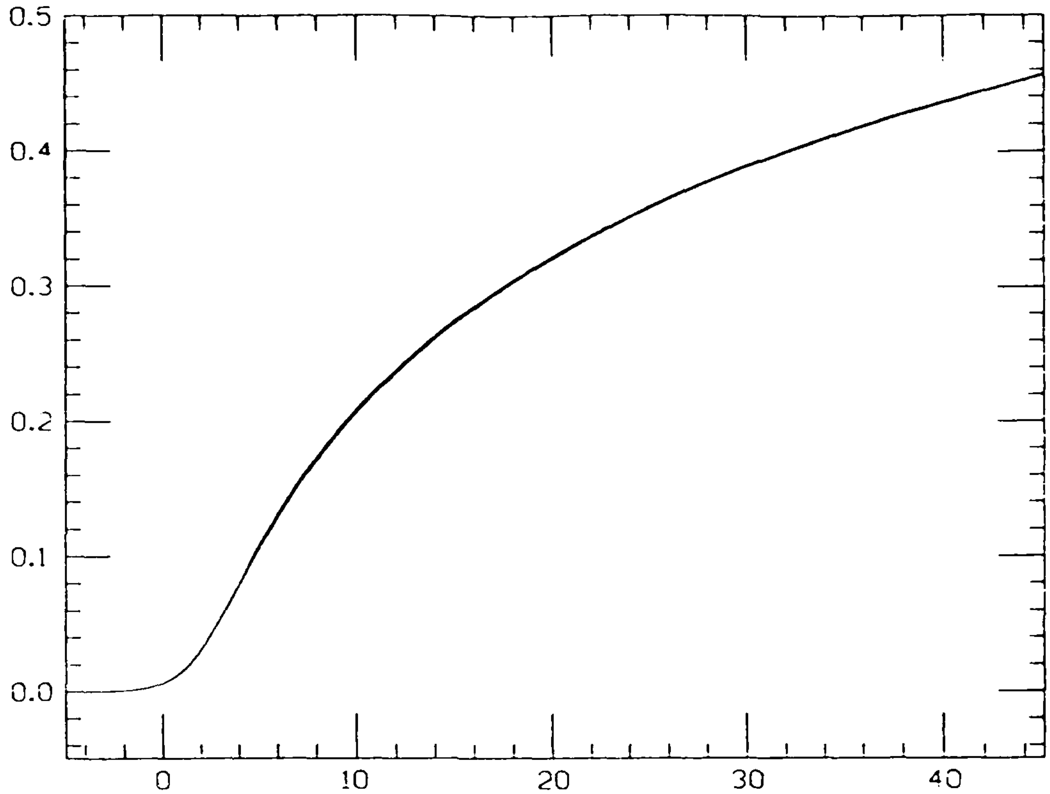

Figure 6.6. The 50-th upper bound and lower bound iterates for $p = 1/10$. The ordinate variable is $\tilde{u}(\eta) = \overline{u}(\eta - 5)$. The abscissa represents the non-dimensional time η.

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